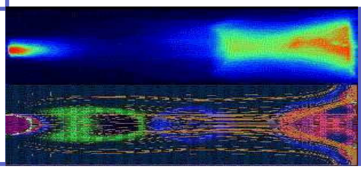
Introduction to BEM

Dr Jasmin Smajic
ABB Corporate Research
Baden - Dättwil
jasmin.smajic@ch.abb.com



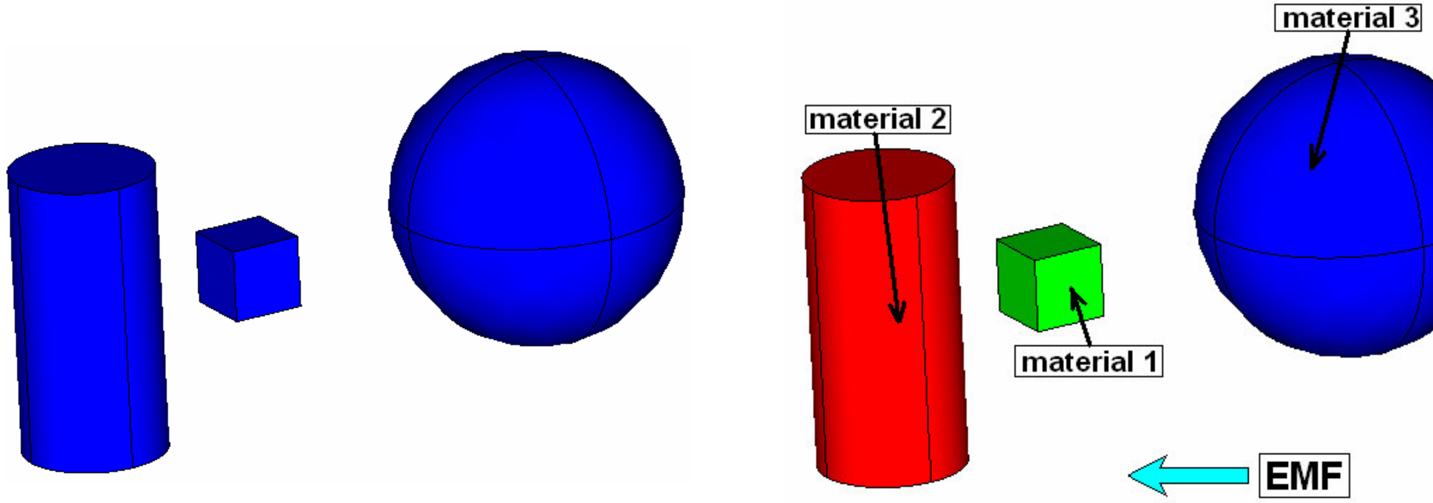
Eidgenössische Technische Hochschule Zürich

Outline

- Introduction to the basic idea of BEM
- Integral formulation of 3D Laplace problem
- BEM treatment of 3D Laplace problem
- 3D electrostatic analysis example
- Matrix compression (Fast multiple technique)
- Singular integrals
- Integral formulation of 3D eddy-currents problem
- 3D eddy-currents analysis example
- Conclusions

Eidgenössische Technische Hochschule Zürich

Introduction to the basic idea of BEM



Mathematical description

$$\left(I\vec{f}\right)(\vec{x}) = 0,\ \vec{x} \in \partial\Omega$$

Model parameters

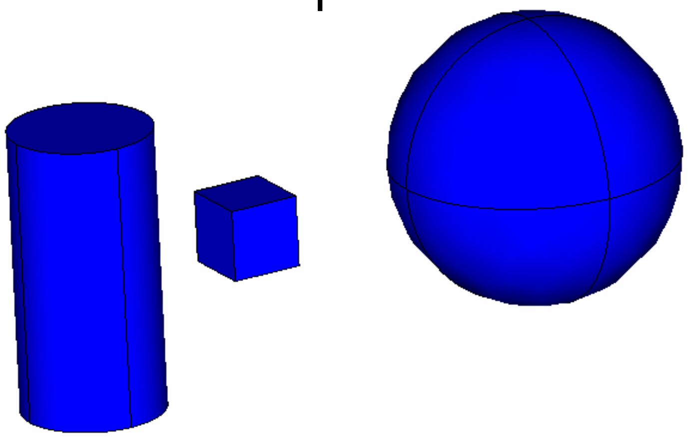
$$material : (\mu, \sigma, \varepsilon)$$

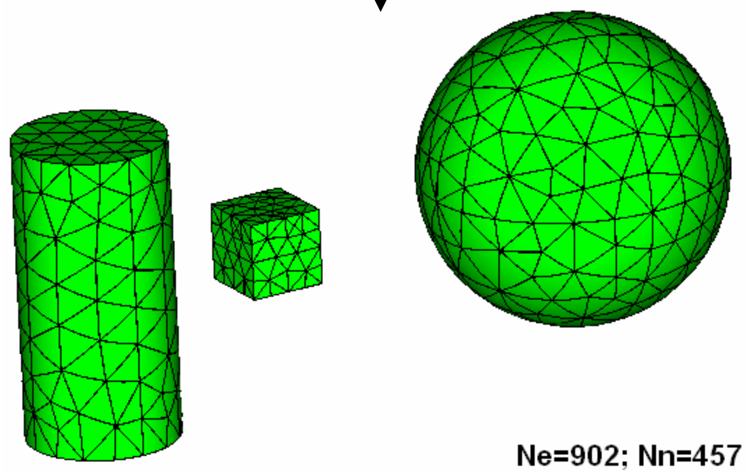
$$frequency : \quad (f)$$

$$sources : (\vec{E}_s, \vec{H}_s, \vec{J}_s)$$

Introduction to the basic idea of BEM

Model discretization





Integral formulation

$$\left(I\vec{f}\right)(\vec{x}) = 0,\ \vec{x} \in \partial\Omega$$

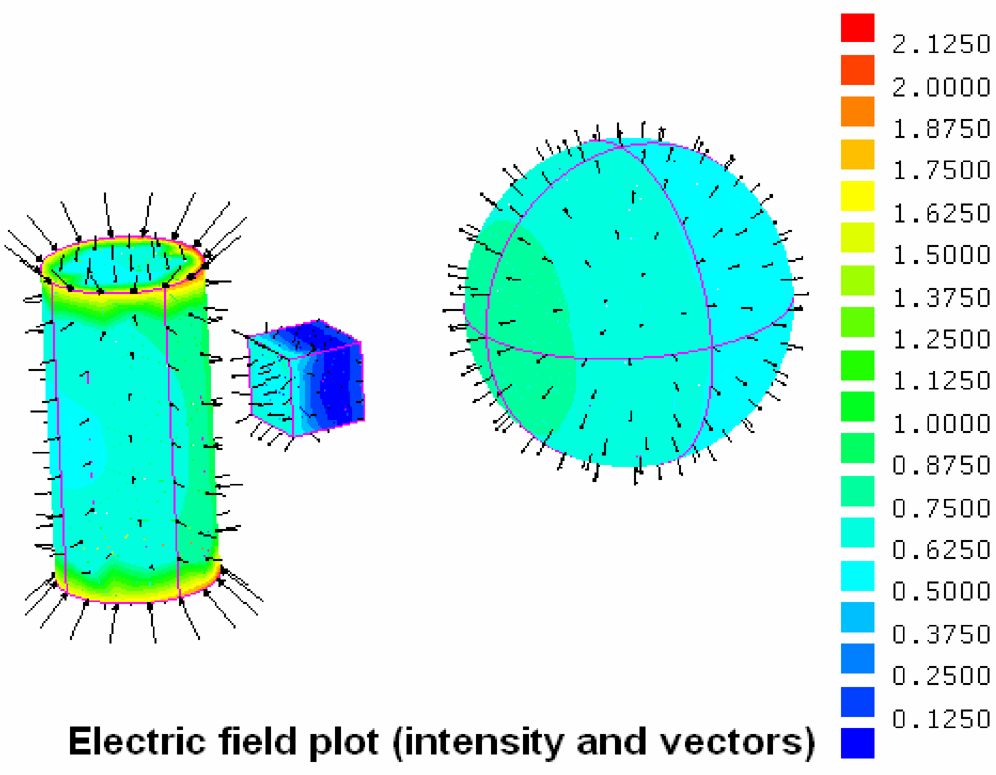
BEM scheme

Large dense linear system of equations

$$[A]\{x\} = \{b\}$$

Eidgenössische Technische Hochschule Zürich

Introduction to the basic idea of BEM



Electric field plot (intensity and vectors)

"Fancy" pictures as a final result of field calculation

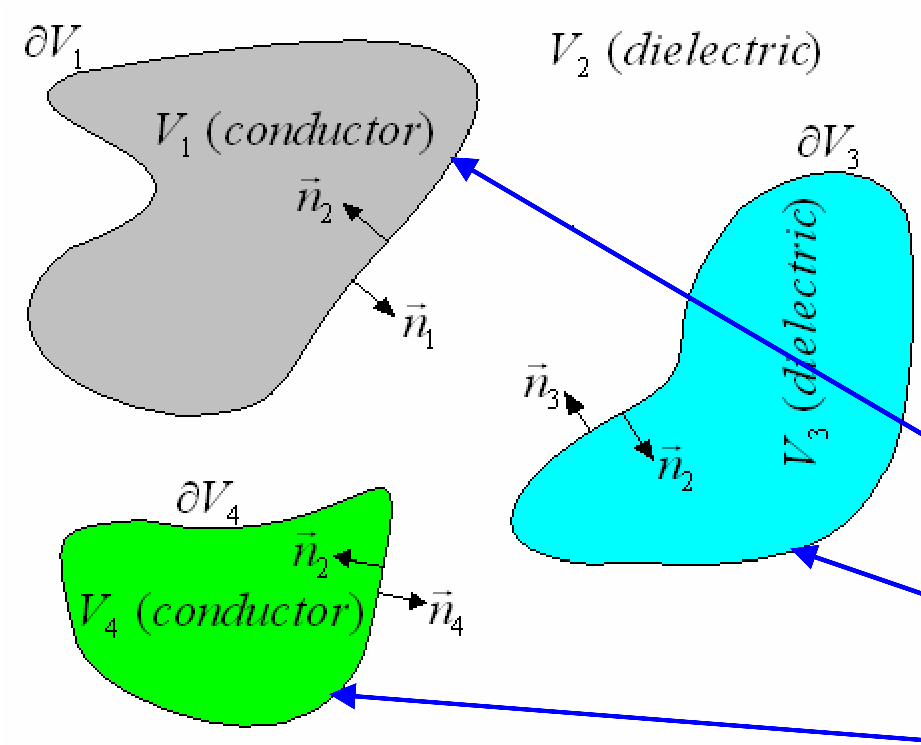
→ **Magnetic, electric field, potential…**

→ **Electromagnetic energy, force …**

→ **Temperature, pressure, velocity…**

Eidgenössische Technische Hochschule Zürich

Integral formulation of 3D Laplace problem



Materials:

1. **Perfect electric conductors (fixed potential electrode -FIXE)**
2. **Perfect electric conductors (floating potential electrode - FLOE)**
3. **Linear homogenous dielectrics**

$$\Delta_y \varphi_k(y) = 0,\ y \in V_k \subseteq R^3$$

$$\varphi_1(x) = \varphi_2(x) = V_0,\ x \in \partial V_1$$

$$\varepsilon_2 \frac{\partial \varphi_2}{\partial n_2}(x) = -\varepsilon_3 \frac{\partial \varphi_3}{\partial n_3}(x),\ x \in \partial V_3$$

$$\nabla \varphi_2(x) \times \vec{n}_4 = 0,\ x \in \partial V_4$$

3D electrostatic analysis

Integral formulation of 3D Laplace problem

$$\Delta_y \varphi_k(y) = 0,\ y \in V_k \subseteq R^3$$

$$-\Delta_y G(x,y) = \delta(x,y),\ y \in V_k \subseteq R^3,\ x \in R^3$$

$$G(x,y) = \frac{1}{4\pi}\frac{1}{|x-y|}$$

Green's function

$$G(x,y)\cdot\Delta_y\varphi_k(y) - \varphi_k(y)\cdot\Delta_y G(x,y) = \delta(x,y)\cdot\varphi_k(y)\ \Big|\ \iiint_{(V_k)}$$

$$\iiint_{(V_k)}\left[G(x,y)\cdot\Delta_y\varphi_k(y) - \varphi_k(y)\cdot\Delta_y G(x,y)\right]dV = \iiint_{(V_k)}\delta(x,y)\cdot\varphi_k(y)dV$$

$$\iiint_{(V_k)}\delta(x,y)\cdot\varphi_k(y)dV = \frac{\theta(x)}{4\pi}\varphi_k(x)$$

$$\theta(x) = \begin{cases} 4\pi,\ x \in V_k - \partial V_k \\ 2\pi,\ x \in \partial V_k,\ if\ \partial V_k\ is\ smooth \\ 0,\ x \notin V_k \end{cases}$$

Eidgenössische Technische Hochschule Zürich

Integral formulation of 3D Laplace problem

$$\iiint_{(V_k)} \left\{ \nabla_y \left[G(x,y) \cdot \nabla_y \varphi_k(y) \right] - \nabla_y \left[\varphi_k(y) \cdot \nabla_y G(x,y) \right] \right\} dV = \frac{\theta(x)}{4\pi} \varphi_k(x)$$

Volume integration

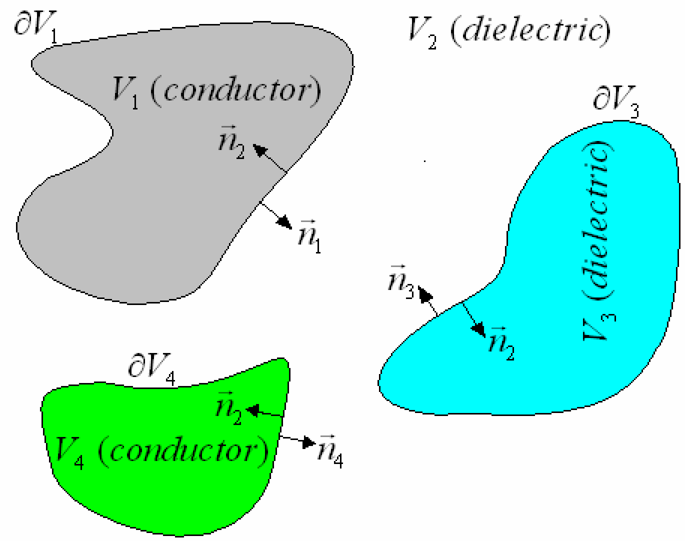
$$\left(Green's\ theorem \right) \Rightarrow$$

$$\oiint_{\partial V_k} \left[G(x,y) \cdot \nabla_y \varphi_k(y) \right] \cdot \vec{n}_y dS_y - \oiint_{(\partial V_k)} \left[\varphi_k(y) \cdot \nabla_y G(x,y) \right] \cdot \vec{n}_y dS_y = \frac{\theta(x)}{4\pi} \varphi_k(x)$$

$$\oiint_{\partial V_k} G(x,y) \cdot \frac{\partial \varphi_k}{\partial n_y}(y)\, dS_y - \oiint_{(\partial V_k)} \varphi_k(y) \cdot \frac{\partial G}{\partial n_y}(x,y)\, dS_y = \frac{\theta(x)}{4\pi} \varphi_k(x)$$

Integration over boundary

Integral formulation of 3D Laplace problem



$x \in R^3$

$$\oiint_{(\partial V_1)} G(x,y) \cdot \frac{\partial \varphi_1}{\partial n_{1y}}(y)\, dS_y - \oiint_{(\partial V_1)} \varphi_1(y) \cdot \frac{\partial G}{\partial n_{1y}}(x,y)\, dS_y = \frac{\theta_1(x)}{4\pi} \varphi_1(x)$$

$$\oiint_{(\partial V_2)} G(x,y) \cdot \frac{\partial \varphi_2}{\partial n_{2y}}(y)\, dS_y - \oiint_{(\partial V_2)} \varphi_2(y) \cdot \frac{\partial G}{\partial n_{2y}}(x,y)\, dS_y = \frac{\theta_2(x)}{4\pi} \varphi_2(x)$$

$$\oiint_{(\partial V_3)} G(x,y) \cdot \frac{\partial \varphi_3}{\partial n_{3y}}(y)\, dS_y - \oiint_{(\partial V_3)} \varphi_3(y) \cdot \frac{\partial G}{\partial n_{3y}}(x,y)\, dS_y = \frac{\theta_3(x)}{4\pi} \varphi_3(x)$$

$$\oiint_{(\partial V_4)} G(x,y) \cdot \frac{\partial \varphi_4}{\partial n_{4y}}(y)\, dS_y - \oiint_{(\partial V_4)} \varphi_4(y) \cdot \frac{\partial G}{\partial n_{4y}}(x,y)\, dS_y = \frac{\theta_4(x)}{4\pi} \varphi_4(x)$$

Integral formulation of 3D Laplace problem

$$\oiint_{(\partial V_1)} G(x,y)\cdot\left[\frac{\partial\varphi_1}{\partial n_{1y}}(y)-\frac{\partial\varphi_2}{\partial n_{1y}}(y)\right]dS_y-\oiint_{(\partial V_1)}[\varphi_1(y)-\varphi_2(y)]\cdot\frac{\partial G}{\partial n_{1y}}(x,y)\,dS_y+$$

$$+\oiint_{(\partial V_3)} G(x,y)\cdot\left[\frac{\partial\varphi_3}{\partial n_{3y}}(y)-\frac{\partial\varphi_2}{\partial n_{3y}}(y)\right]dS_y-\oiint_{(\partial V_3)}[\varphi_3(y)-\varphi_2(y)]\cdot\frac{\partial G}{\partial n_{3y}}(x,y)\,dS_y+$$

$$+\oiint_{(\partial V_4)} G(x,y)\cdot\left[\frac{\partial\varphi_4}{\partial n_{4y}}(y)-\frac{\partial\varphi_2}{\partial n_{4y}}(y)\right]dS_y-\oiint_{(\partial V_4)}[\varphi_4(y)-\varphi_2(y)]\cdot\frac{\partial G}{\partial n_{4y}}(x,y)\,dS_y$$

$$=\frac{\theta_1(x)}{2\pi}\varphi_1(x)+\frac{\theta_2(x)}{2\pi}\varphi_2(x)+\frac{\theta_3(x)}{2\pi}\varphi_3(x)+\frac{\theta_4(x)}{2\pi}\varphi_4(x)$$

↓

$$-\oiint_{(\partial V_1)} G(x,y)\cdot\frac{\partial\varphi_2}{\partial n_{1y}}(y)\,dS_y+\oiint_{(\partial V_3)} G(x,y)\cdot\left[\frac{\partial\varphi_3}{\partial n_{3y}}(y)-\frac{\partial\varphi_2}{\partial n_{3y}}(y)\right]dS_y+$$

$$-\oiint_{(\partial V_4)} G(x,y)\cdot\frac{\partial\varphi_2}{\partial n_{4y}}(y)\,dS_y=\frac{\theta_1(x)}{2\pi}\varphi_1(x)+\frac{\theta_2(x)}{2\pi}\varphi_2(x)+\frac{\theta_3(x)}{2\pi}\varphi_3(x)+\frac{\theta_4(x)}{2\pi}\varphi_4(x)$$

(*)

Eidgenössische Technische Hochschule Zürich

Integral formulation of 3D Laplace problem

BEM is related to the field sources over the boundary

$$y \in \partial V_1 \Rightarrow \varepsilon_2 \frac{\partial \varphi_2}{\partial n_{1y}}(y) = -\sigma_{1f}(y), \sigma_1'(y) = \sigma_{1f}(y) + \sigma_{1pol}(y) = -\varepsilon_0 \frac{\partial \varphi_2}{\partial n_{1y}}(y), \sigma_1(y) = \frac{\sigma_1'(y)}{\varepsilon_0}$$

$$y \in \partial V_3 \Rightarrow \frac{\partial \varphi_3}{\partial n_{3y}}(y) - \frac{\partial \varphi_2}{\partial n_{3y}}(y) = \left(1 - \frac{\varepsilon_2}{\varepsilon_3}\right) \frac{\partial \varphi_2}{\partial n_{3y}}(y) = \frac{\sigma_3'(y)}{\varepsilon_0}, \sigma_3(y) = \frac{\sigma_3'(y)}{\varepsilon_0}$$

$$y \in \partial V_4 \Rightarrow \varepsilon_2 \frac{\partial \varphi_2}{\partial n_{4y}}(y) = -\sigma_{4f}(y), \sigma_4'(y) = \sigma_{1f}(y) + \sigma_{1pol}(y) = -\varepsilon_0 \frac{\partial \varphi_2}{\partial n_{4y}}(y), \sigma_4(y) = \frac{\sigma_4'(y)}{\varepsilon_0}$$

Our integral formulation deals with a total surface charge density !

Total charge = Free charge (conductor) + Polarized charge (dialect. interface)

Integral formulation of 3D Laplace problem

$$x \in \partial V_1 \overset{(*)}{\Rightarrow}$$

$$\oiint_{(\partial V_1)} G(x,y)\cdot\sigma_1(y)\,dS_y + \oiint_{(\partial V_3)} G(x,y)\cdot\sigma_3(y)\,dS_y + \oiint_{(\partial V_4)} G(x,y)\cdot\sigma_4(y)\,dS_y = V_0$$

OK!

$$x \in \partial V_3 \overset{(*)}{\Rightarrow}$$

$$\oiint_{(\partial V_1)} G(x,y)\cdot\sigma_1(y)\,dS_y + \oiint_{(\partial V_3)} G(x,y)\cdot\sigma_3(y)\,dS_y + \oiint_{(\partial V_4)} G(x,y)\cdot\sigma_4(y)\,dS_y = \frac{\theta_2(x)}{4\pi}\varphi_2(x) + \frac{\theta_3(x)}{4\pi}\varphi_3(x)$$

?!!

Problem: potential and charge density are unknowns (in the same time)!

Solution:

$$\oiint_{(\partial V_1)} G(x,y)\cdot\sigma_1(y)\,dS_y + \oiint_{(\partial V_3)} G(x,y)\cdot\sigma_3(y)\,dS_y + \oiint_{(\partial V_4)} G(x,y)\cdot\sigma_4(y)\,dS_y = \frac{\theta_2(x)}{4\pi}\varphi_2(x) + \frac{\theta_3(x)}{4\pi}\varphi_3(x) \;\Big|\; \frac{\partial}{\partial n_{3x}}$$

Integral formulation of 3D Laplace problem

$x \in \partial V_3 \Rightarrow$

Dielectrics interface

$$\oiint\limits_{(\partial V_1)} \frac{\partial G}{\partial n_{3x}}(x,y)\cdot\sigma_1(y)\,dS_y + \oiint\limits_{(\partial V_3)} \frac{\partial G}{\partial n_{3x}}(x,y)\cdot\sigma_3(y)\,dS_y + \oiint\limits_{(\partial V_4)} \frac{\partial G}{\partial n_{3x}}(x,y)\cdot\sigma_4(y)\,dS_y =$$

$$= \frac{1}{2}\frac{\varepsilon_3+\varepsilon_2}{\varepsilon_3-\varepsilon_2}\sigma_3(x)$$

$x \in \partial V_4 \Rightarrow$

Floating potential electrode

$$\oiint\limits_{(\partial V_1)} \frac{\partial G}{\partial n_{4x}}(x,y)\cdot\sigma_1(y)\,dS_y + \oiint\limits_{(\partial V_3)} \frac{\partial G}{\partial n_{4x}}(x,y)\cdot\sigma_3(y)\,dS_y + \oiint\limits_{(\partial V_4)} \frac{\partial G}{\partial n_{4x}}(x,y)\cdot\sigma_4(y)\,dS_y =$$

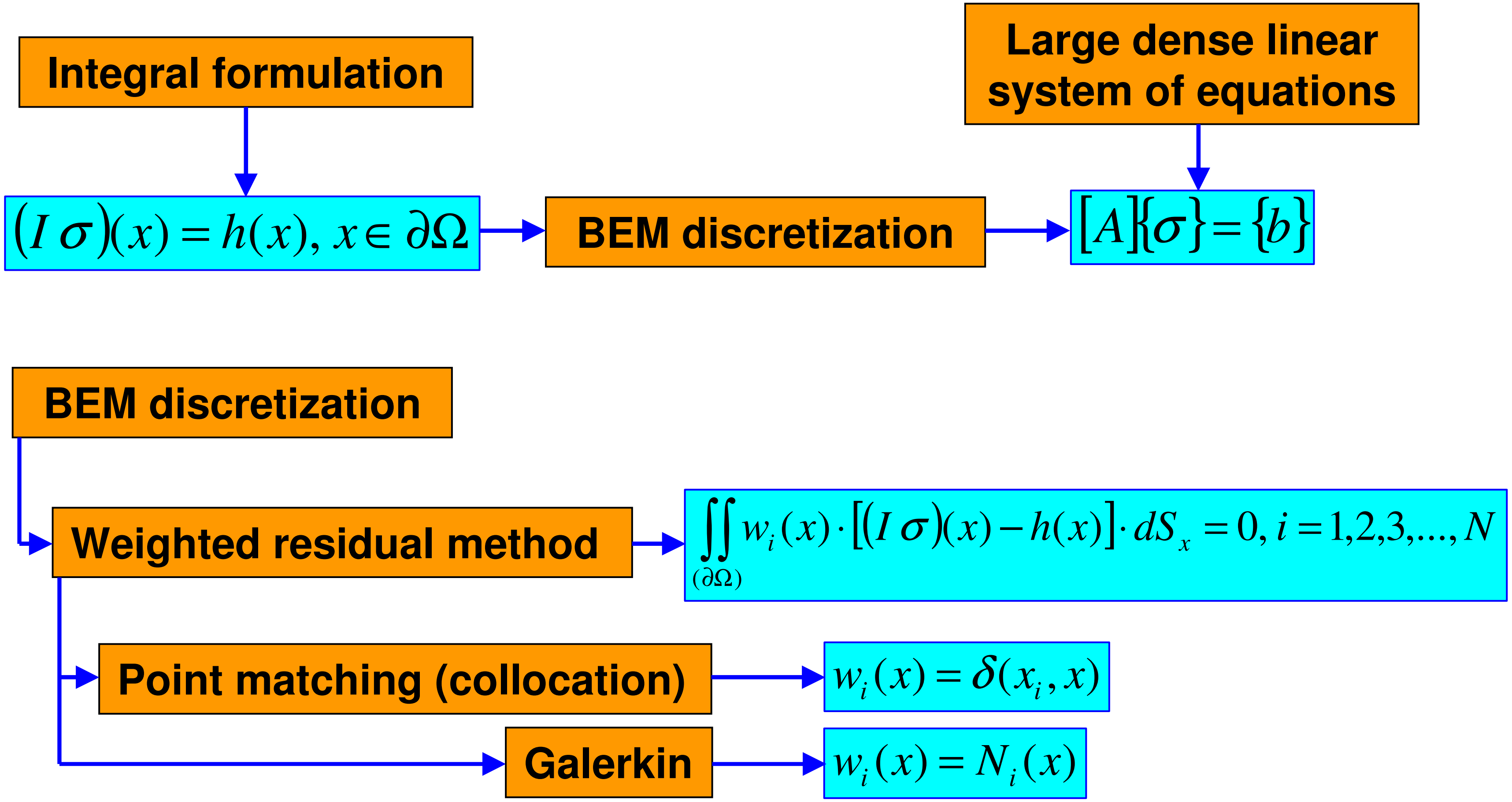
$$= -\frac{1}{2}\sigma_4(x)$$

$x \in \partial V_1 \Rightarrow$

Fixed potential electrode

$$\oiint\limits_{(\partial V_1)} G(x,y)\cdot\sigma_1(y)\,dS_y + \oiint\limits_{(\partial V_3)} G(x,y)\cdot\sigma_3(y)\,dS_y + \oiint\limits_{(\partial V_4)} G(x,y)\cdot\sigma_4(y)\,dS_y = V_0$$

BEM treatment of 3D Laplace problem



Eidgenössische Technische Hochschule Zürich

BEM treatment of 3D Laplace problem

Point matching (collocation)

$$w_i(x) = \delta(x_i, x)$$

$$\left(I\,\sigma\right)(x_i) = h(x_i);\ i = 1,2,3,\ldots,N;\ x_i \in \partial\Omega$$

Linear nodal triangular element

$$\sigma^e(x) = \sum_{j=1}^{3} N_j^e(x)\cdot\sigma_j^e,\ x \in \Delta^e$$

Concept of elemental contribution

$$\left(F\sigma\right)(x_i) = \left(I\,\sigma\right)(x_i) - h(x_i)$$

$$\left(F\sigma\right)(x) = \sum_{e=1}^{N_e}\left(F^e\sigma^e\right)(x) = \sum_{e=1}^{N^e}\left(\left[K^e\right]\left\{\sigma^e\right\} - \left\{b^e\right\}\right)$$

$$K_{ij}^e = \begin{cases} \iint\limits_{(\Delta^e)}\left(G(x_i, y)\cdot N_j^e(y)\right)d\Omega,\ \Delta^e \in \partial V_1 \\ C + \iint\limits_{(\Delta^e)}\left(\dfrac{\partial G}{\partial n_y}(x_i, y)\cdot N_j^e(y)\right)d\Omega,\ \Delta^e \in \partial V_{3,4} \end{cases}$$

$$b_i^e = \begin{cases} V_0,\ x_i \in \partial V_1 \\ 0,\ otherwise \end{cases}$$

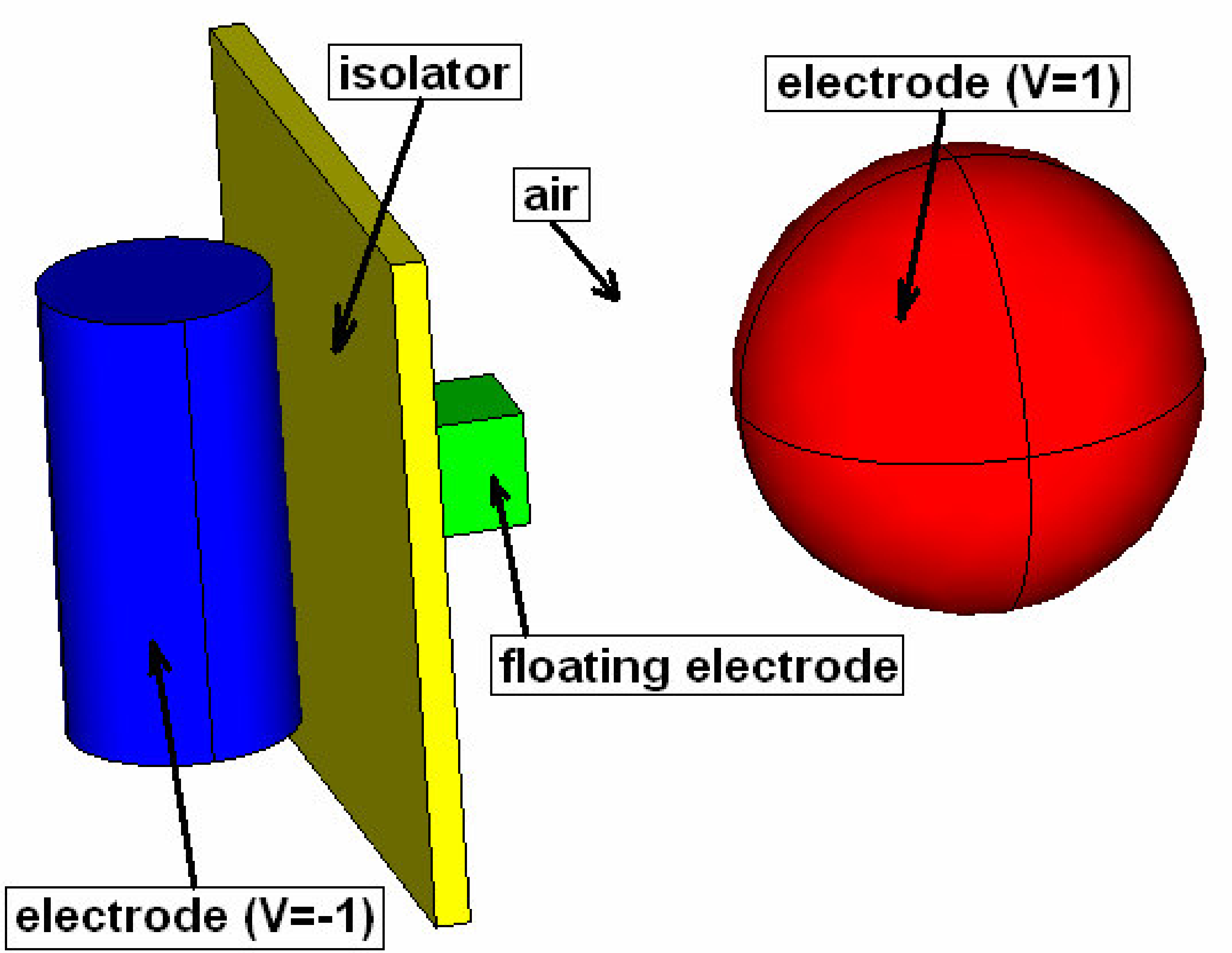
$$i, j = 1,2,3$$

BEM treatment of 3D Laplace problem

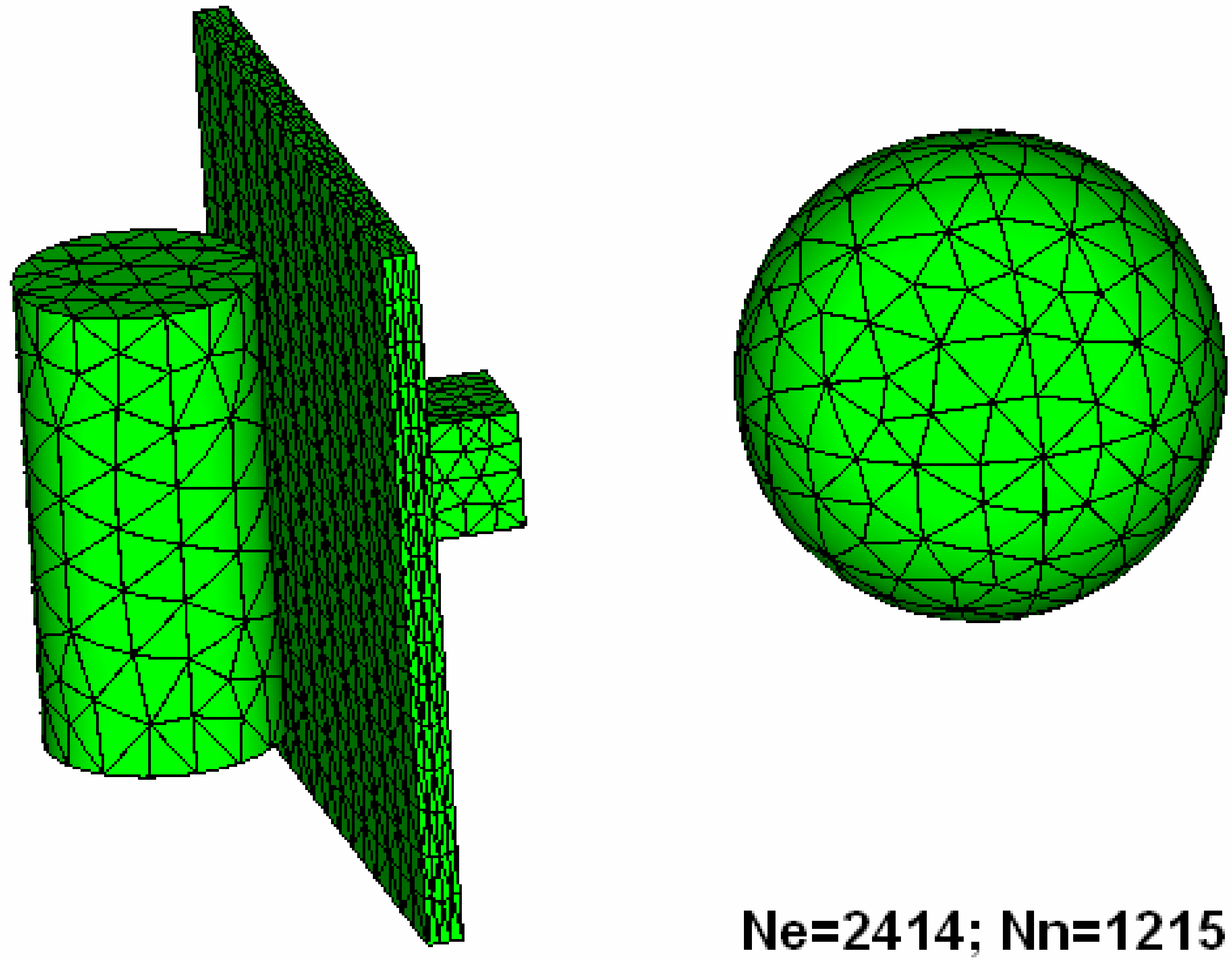
Large dense linear system of equations → $[A]\{\sigma\}=\{b\}$

- Iterative solver has to be used
- BiCG, GMRES
- Asymmetric matrix has been produced
- Preconditioning of the matrix is very important
- Matrix compression is necessary
- Fast multipole algorithm (improvement for matrix compression and solution speed)

3D electrostatic example



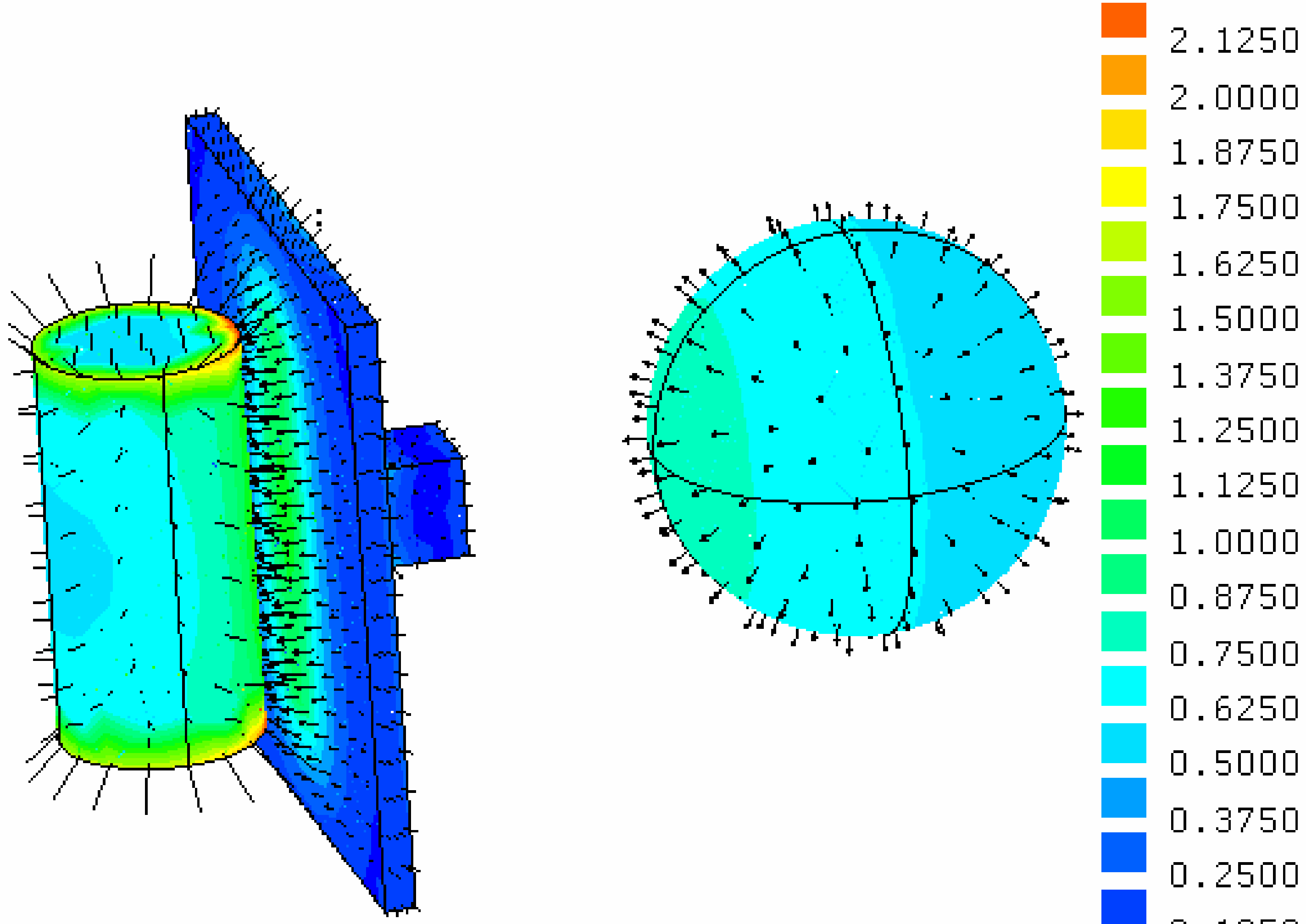
3D electrostatic example



Ne=2414; Nn=1215

Eidgenössische Technische Hochschule Zürich

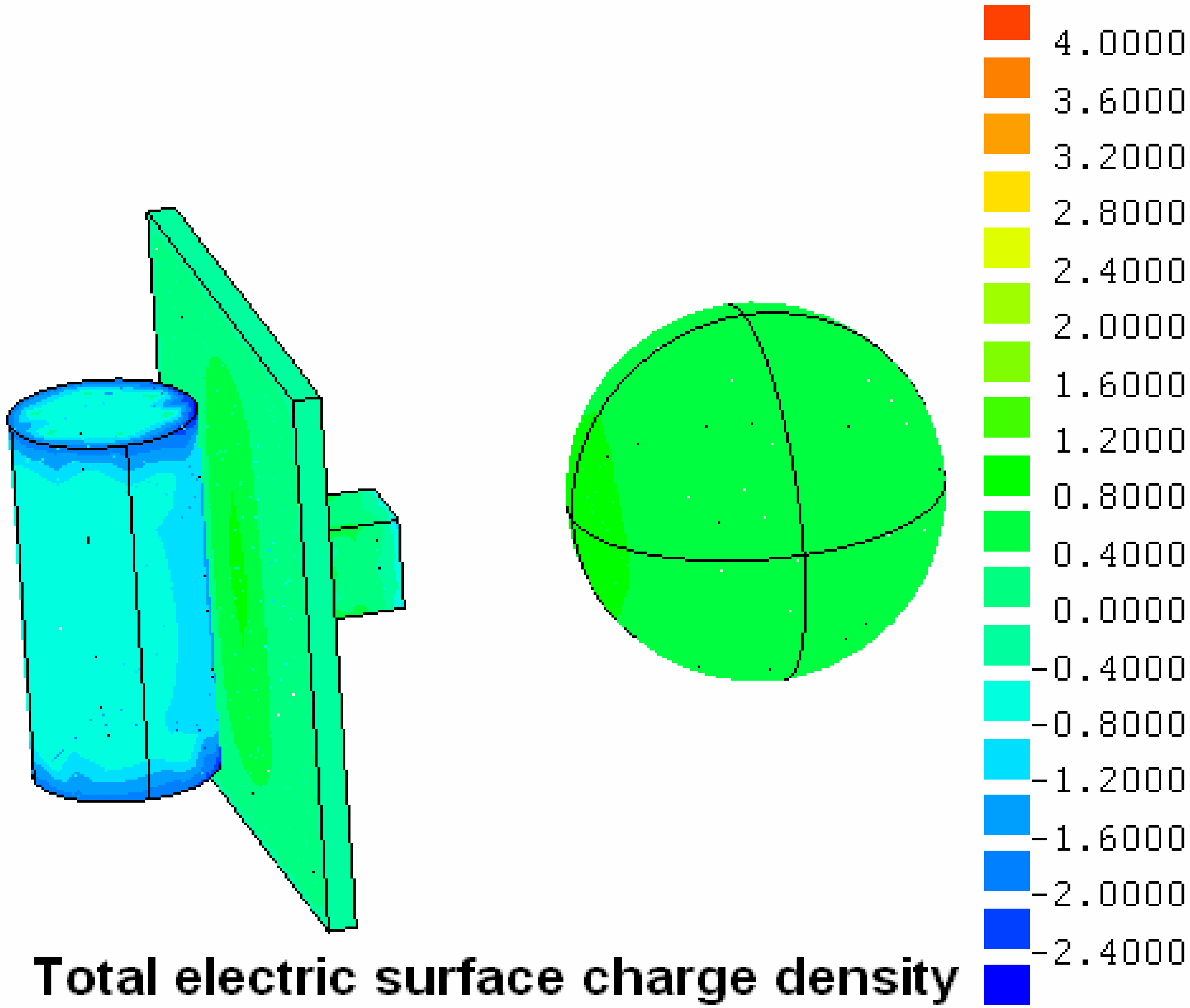
3D electrostatic example



Electric field (intensity and vectors)

Eidgenössische Technische Hochschule Zürich

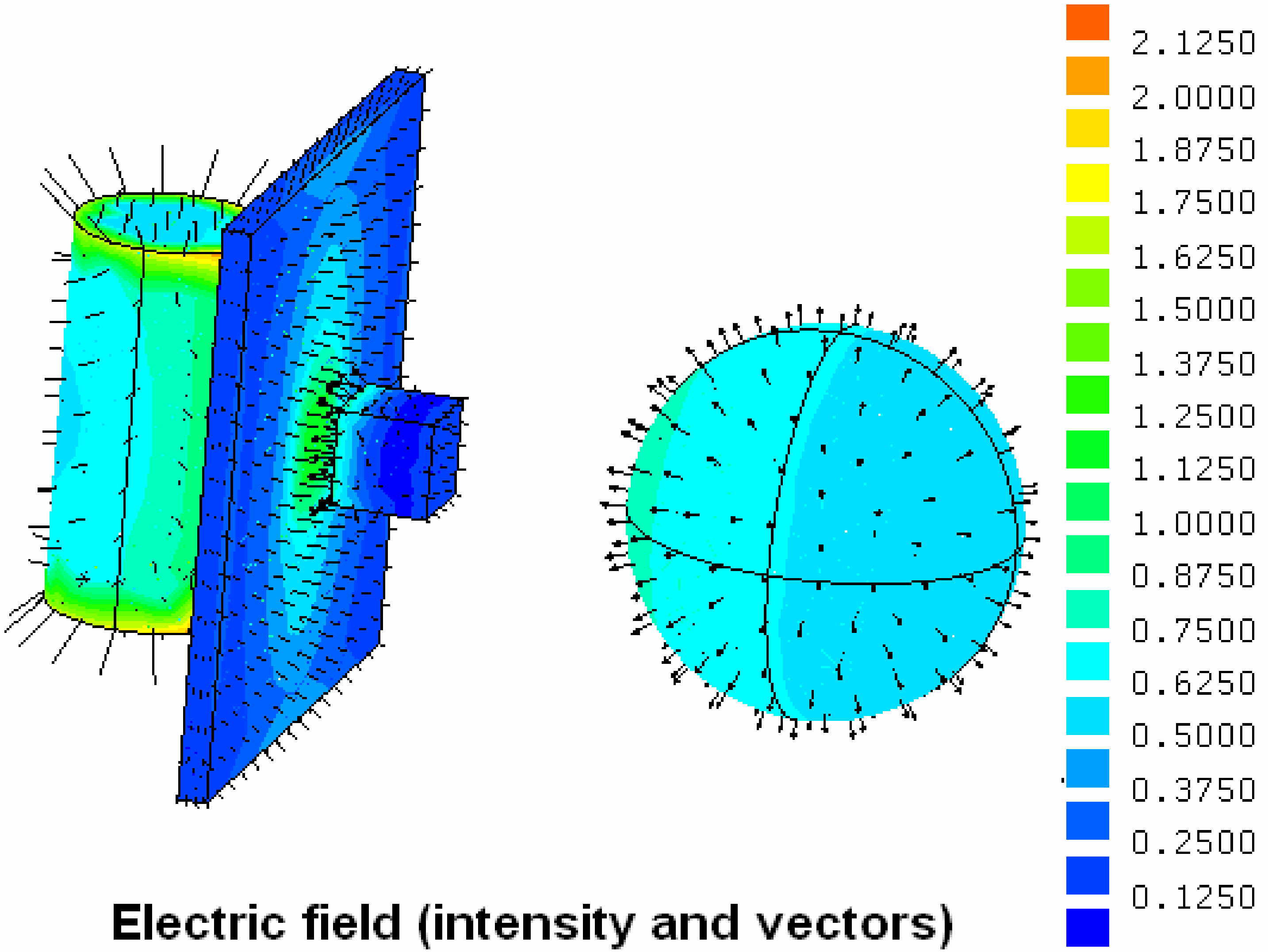
3D electrostatic example



Total electric surface charge density

Eidgenössische Technische Hochschule Zürich

3D electrostatic example

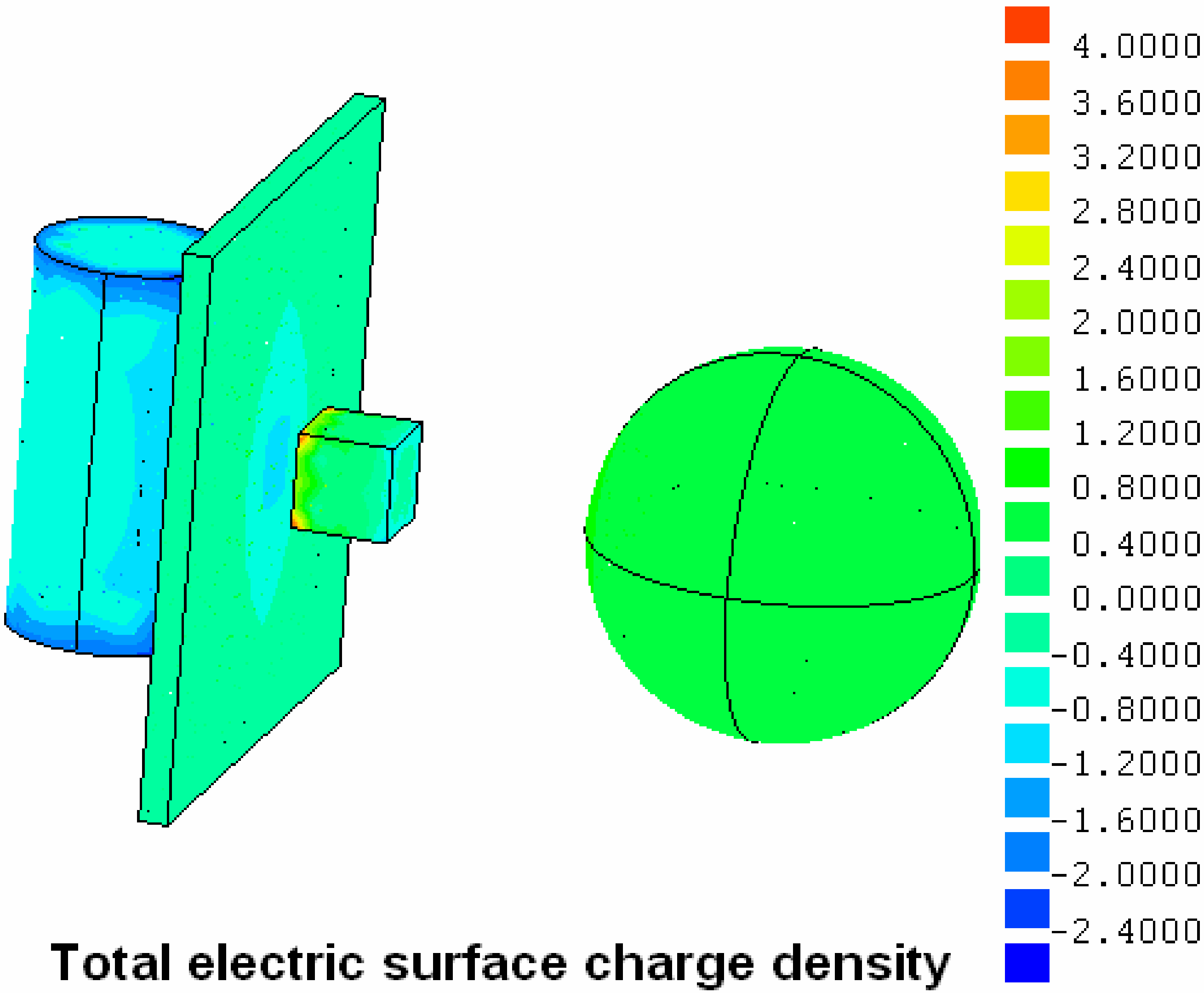


Electric field (intensity and vectors)

ETH Eidgenössische Technische Hochschule Zürich

ABB

3D electrostatic example



Total electric surface charge density

Eidgenössische Technische Hochschule Zürich

3D electrostatic example

Basic solution data

- Nn=1215; Ne=2414
- Full matrix size = 1215 x 1215
- Full matrix memory = 17MB
- CPU time = 60 sec (matrix assemblage + solution + field calc.)
- CPU time (matrix generation) = 32 sec
- Diagonal preconditioning + GMRES (convergence in 21 iterat.)
- CPU time (system solution) = 1 sec
- CPU time (field calculation) = 27 sec

Matrix compression (fast multipole technique)

Kernel expansion

$$|x-y| >> 0\,(farfield) \Rightarrow$$

$$K(x,y) \approx K_m(x,y;x_0,y_0) = \sum_{(\mu,\nu)\in I_m} K_{(\mu,\nu)}(x_0,y_0)\cdot X_\mu(x,x_0)\cdot Y_\nu(y,y_0)$$

Taylor-, Multipole-, Cebysev- expansion

$$|x-x_0| + |y-y_0| \le \eta \cdot |x_0 - y_0|$$

Farfield condition

GMRES with clustering

$$v = \tilde{A}\cdot u = N\cdot u + \sum_{(\sigma,\tau)\in F} X_\sigma^T\left(F_{\sigma\tau}\left(Y_\tau \cdot u\right)\right)$$

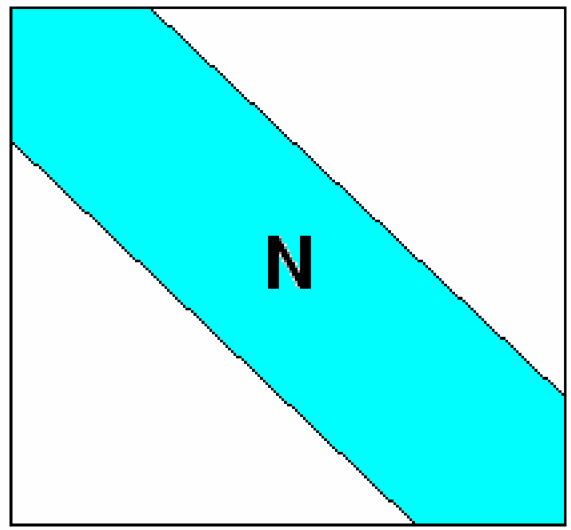
Matrix-vector multiplication

ETH Eidgenössische Technische Hochschule Zürich

ABB

Matrix compression (fast multipole technique)

$\tilde{A} =$



GMRES with clustering

$$v = \tilde{A} \cdot u = N \cdot u + \sum_{(\sigma,\tau)\in F} X_\sigma^T \left(F_{\sigma\tau} \left(Y_\tau \cdot u \right) \right)$$

Matrix-vector multiplication

Matrix compression (fast multipole technique)

Basic solution data (with compression)

- Nn=1215; Ne=2414
- Full matrix size = 1215 x 1215
- Near-field matrix memory = 1MB
- CPU time = 22 sec (matrix assemblage + solution + field calc.)
- CPU time (matrix generation) = 10 sec
- Diagonal preconditioning + GMRES (convergence in 21 iterat.)
- CPU time (system solution) = 1 sec
- CPU time (field calculation) = 11 sec

Singular integrals

Point collocation

$$\iint\limits_{(\Delta^e)} \left(K(x_i, y) \cdot N_j^e(y) \right) d\Omega, \; x_i \in \Delta^e$$

CPV integrals

$$K(x,y) \rightarrow \frac{1}{|x-y|^n}$$

$$x \rightarrow y$$

n = 1 – weak singularity

n = 2 – strong singularity

n = 3 – hyper singularity

Flat elements: analytical integration

Curved elements:

- Numerical integration (Gauss quadrature)
- Coordinate transformations (Duffy transformation)

Integral formulation of 3D eddy-currents problem

3D eddy-current problem (at industrial frequencies)

$\nabla \times \vec{H} = \vec{J}$ $\nabla \times \vec{E} = -j\omega\vec{B}$ ← **curl Maxwell equations**

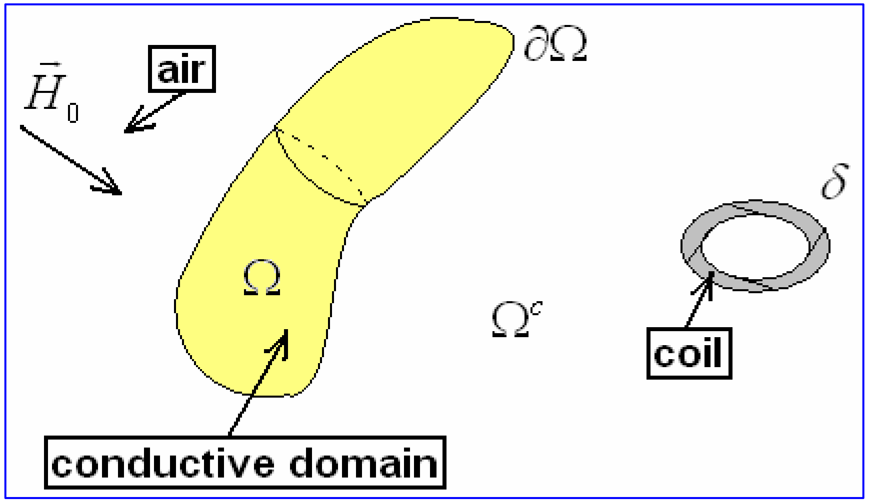
$\vec{B} = \mu_0 \mu_r \vec{H}$ $J = \sigma\vec{E}$ ← **Constitutive relations**

$\nabla \cdot \vec{B} = 0$ $\nabla \cdot \vec{J} = 0$ ← **div Maxwell equations**

$\vec{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{K}$ $\vec{n} \times \left(\vec{E}_1 - \vec{E}_2\right) = 0$ ← **Interface conditions**

Integral formulation of 3D eddy-currents problem

$\vec{H} - \varphi \quad formulation$



$\vec{H}_0$ – impressed field

$\vec{H}_\delta$ – field due to the coil

$\vec{H}_S$ – secondary mag. field

$$\vec{H}^{\Omega^c} = \vec{H}_0 + \vec{H}_\delta + \vec{H}_S$$

$$\vec{H}_S = -\nabla\varphi$$

$$\nabla\times\nabla\times\vec{H}^{\Omega} = -j\omega\mu\sigma\vec{H}^{\Omega}$$

$$\nabla\cdot\vec{H}^{\Omega} = 0 \qquad \nabla^2\cdot\varphi = 0$$

Interface conditions

$$\vec{n}\times\left(\vec{H}^{\Omega} + \nabla\varphi\right) = \vec{n}\times\left(\vec{H}_\delta + \vec{H}_0\right)$$

$$\vec{n}\cdot\left(\mu\vec{H}^{\Omega} + \mu_0\nabla\varphi\right) = \mu_0\vec{n}\cdot\left(\vec{H}_\delta + \vec{H}_0\right)$$

Integral formulation of 3D eddy-currents problem

$\vec{H}-\varphi$ *formulation*

$$-\frac{1}{2}\vec{J}(\xi)+\frac{1}{4\pi}\oiint_{(\partial\Omega)}\left[\vec{n}_\xi\times\left(\vec{J}(\eta)\times\nabla_\xi K(\eta,\xi)\right)\right]dS_\eta -$$
$$-\frac{1}{4\pi}\oiint_{(\partial\Omega)}\left[\sigma_m(\eta)\left(\vec{n}_\xi\times\nabla_\xi G(\eta,\xi)\right)\right]dS_\eta=-\left[\vec{H}_0^t(\xi)+\vec{H}_\delta^t(\xi)\right]$$

$$\mathrm{G}(\eta,\xi)=\frac{1}{\mathrm{r}_{\eta,\xi}}$$

$$\forall\,\eta,\xi\in\partial\Omega$$

$$-\frac{1}{2}\sigma_m(\xi)-\frac{1}{4\pi}\oiint_{(\partial\Omega)}\left[\sigma_m(\eta)\left(\vec{n}_\xi\cdot\nabla_\xi G(\eta,\xi)\right)\right]dS_\eta -$$
$$-\frac{\mu}{4\pi\mu_0}\oiint_{(\partial\Omega)}\left[\vec{n}_\xi\cdot\left(\vec{J}(\eta)\times\nabla_\xi K(\eta,\xi)\right)\right]dS_\eta=-\left[\vec{H}_0^n(\xi)+\vec{H}_\delta^n(\xi)\right]$$

$\vec{J}$ – virtual current

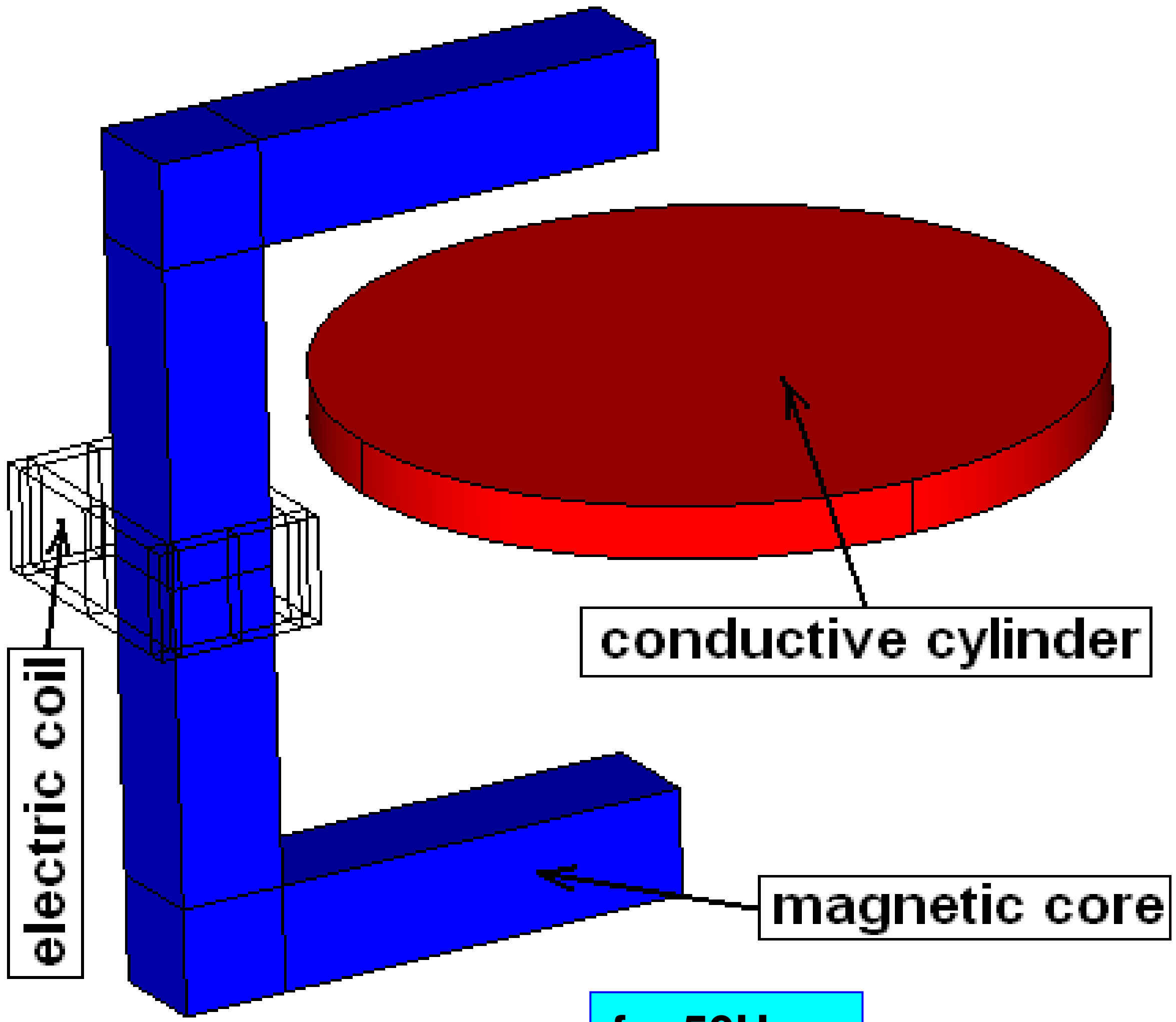
σ_m – virtual magnetic charge

$$\mathrm{K}(\eta,\xi)=\frac{\mathrm{e}^{-(1+j)\cdot k\cdot r_{\eta,\xi}}}{\mathrm{r}_{\eta,\xi}}$$

$$\mathrm{k}=\sqrt{\omega\mu_0\mu_r\sigma/2}$$

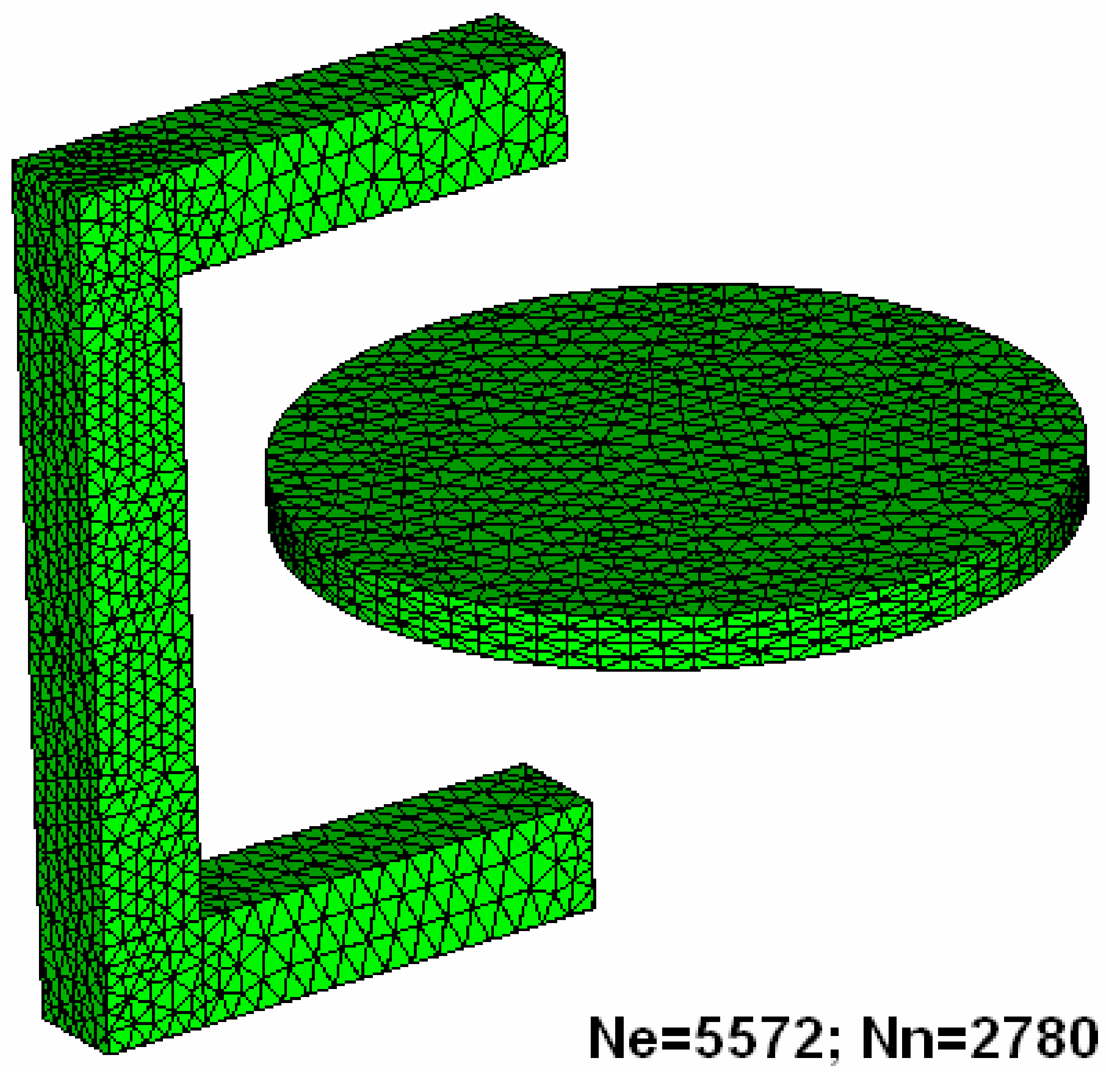
Eidgenössische Technische Hochschule Zürich

3D eddy-currents analysis example

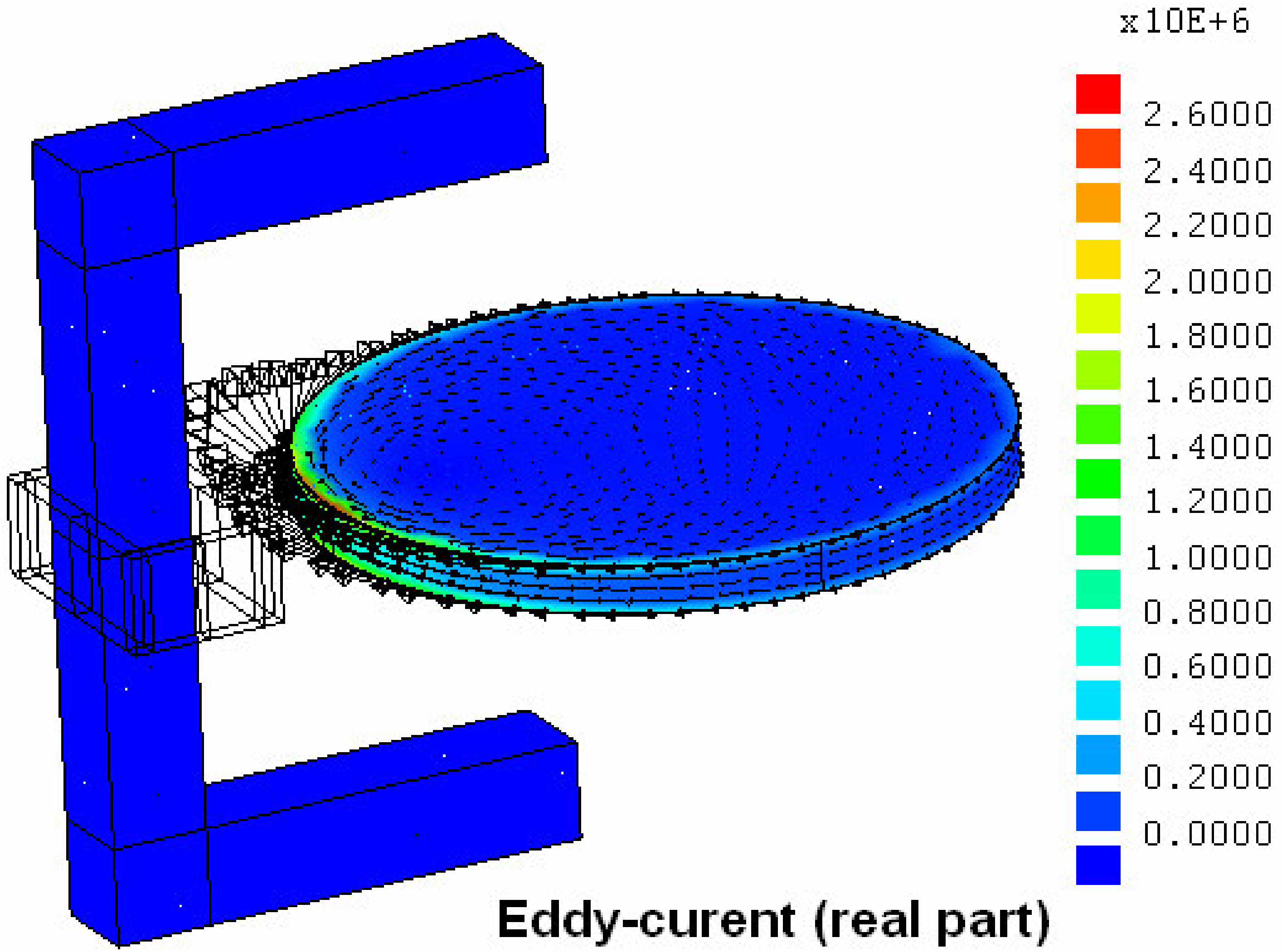


f = 50Hz

3D eddy-currents analysis example

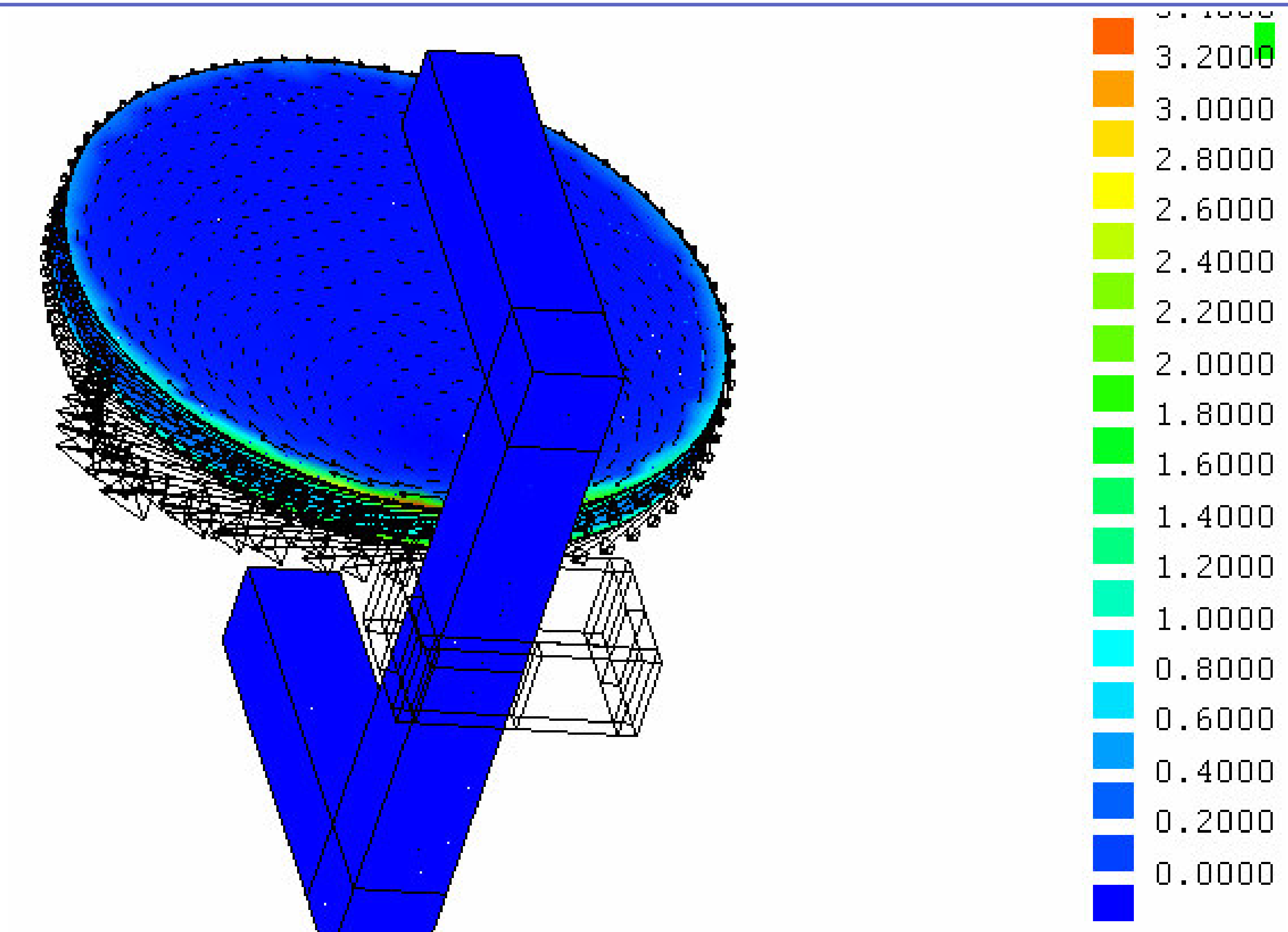


3D eddy-currents analysis example



Eddy-curent (real part)

3D eddy-currents analysis example



Eddy-current (imaginary part)

Eidgenössische Technische Hochschule Zürich

3D eddy-currents analysis example

Basic solution data (with compression)

- Nn=2780; Ne=5552
- Full matrix size = 2780 x 2780
- Near-field matrix memory = 44MB
- CPU time = 100 sec (matrix assemblage + solution + field calc.)
- CPU time (matrix generation) = 47 sec
- Diagonal preconditioning + GMRES (convergence in 54 iterat.)
- CPU time (system solution) = 7 sec
- CPU time (field calculation) = 46 sec

Conclusions

- We have illustrated BEM application to 3D electrostatic and 3D eddy-currents problems
- In the case of homogeneous and linear materials BEM is dominant in comparison by FEM
- Obviously BEM is very powerful simulation tool for certain class of electromagnetic problems
- Dense matrix and singular integrals are the basic drawback of BEM
- Matrix compression techniques (fast multipole method or ACA) are promising techniques for the future of BEM.

ABB