Electromagnetics

Traditional areas

Electrostatics

- Capacitors
- High voltage

Magnetostatics

- Inductors
- Motors / Generators

Quasistatics

- Sensors
- Electronic circuits
- Transmission lines

Electrodynamics

- Antennas
- Wave propagation / scattering
- Gratings
- Guided waves
- Resonators...

«Modern areas»

- Integrated optics
- Optical computers
- Nearfield optics
- Nano optics
- Photonic crystals
- Metamaterials
- Plasmonics

Mixed with other disciplines

- Semiconductors
- Lasers, LED, LCD
- Solar cells, Photovoltaics
- RF-MEMS
- (Nano-) Robots
- Bio/medical...

Maxwell equations

$$\mathrm{curl}\,\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$$

$$\mathrm{curl}\,\vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t)$$

$$\mathrm{div}\,\vec{D}(\vec{r},t) = \rho(\vec{r},t)$$

$$\mathrm{div}\,\vec{B}(\vec{r},t) = 0$$

Four coupled differential equations
More coupling: description of materials (permeability, permittivity, conductivity, etc.
First order derivatives in space (curl, div) and time

What do the various fields mean?
How can we obtain solutions?
Can we derive other types of equations?

Differential operators in 3D space

$$\vec{e}_\ell \cdot \operatorname{grad}\,\phi = \lim_{r\to 0} \frac{\oint_{\partial\ell} \phi \, \mathrm{d}P}{\int_\ell \mathrm{d}\ell}\,, \qquad \vec{e}_n \cdot \operatorname{curl}\,\vec{A} = \lim_{r\to 0} \frac{\oint_{\partial S} \vec{A}\, \mathrm{d}\vec{\ell}}{\int_S \mathrm{d}S}\,, \qquad \operatorname{div}\,\vec{B} = \lim_{r\to 0} \frac{\oint_{\partial V} \vec{B}\, \mathrm{d}\vec{S}}{\int_V \mathrm{d}V}\,,$$

Background: Voltage between 2 points, flux through area, content of a volume

$$U = \int_\ell \vec{v}\, \mathrm{d}\vec{\ell} = \oint_{\partial\ell} \phi\, \mathrm{dP}\,, \qquad \Phi = \int_S \vec{a}\, \mathrm{d}\vec{S} = \oint_{\partial S} \vec{A}\, \mathrm{d}\vec{\ell}\,, \qquad I = \int_V p\, \mathrm{d}V = \oint_{\partial V} \vec{B}\, \mathrm{d}\vec{S}\,.$$

... law, Stokes law Gauss law

$$U = \int_\ell \operatorname{grad}\,\phi\, \mathrm{d}\vec{\ell} = \oint_{\partial\ell} \phi\, \mathrm{d}P\,, \qquad \Phi = \int_S \operatorname{curl}\,\vec{A}\, \mathrm{d}\vec{S} = \oint_{\partial S} \vec{A}\, \mathrm{d}\vec{\ell}\,, \qquad I = \int_V \operatorname{div}\,\vec{B}\, \mathrm{d}V = \oint_{\partial V} \vec{B}\, \mathrm{d}\vec{S}\,.$$

Orthogonal coordinates *u,v,w*, metric coefficients *g*

$$\operatorname{grad}\,\phi = \frac{\phi_{,u}}{g_u}\vec{e}_u + \frac{\phi_{,v}}{g_v}\vec{e}_v + \frac{\phi_{,w}}{g_w}\vec{e}_w\,. \qquad \operatorname{div}\,\vec{B} = \frac{(g_v g_w B_u)_{,u} + (g_w g_u B_v)_{,v} + (g_u g_v B_w)_{,w}}{g_u g_v g_w}\,,$$

$$\operatorname{curl}\,\vec{B} = \frac{(g_w A_w)_{,v} - (g_v A_v)_{,w}}{g_v g_w}\vec{e}_u + \frac{(g_u A_u)_{,w} - (g_w A_w)_{,u}}{g_w g_u}\vec{e}_v + \frac{(g_v A_v)_{,u} - (g_u A_u)_{,v}}{g_u g_v}\vec{e}_w\,,$$

Differential operators in 3D space: Volume integrals

Alternative:

$$\operatorname{div} \vec{B} = \lim_{r \to 0} \frac{\oint_{\partial V} \vec{B} \, d\vec{S}}{\int_V dV}, \qquad \operatorname{curl} \vec{A} = \lim_{r \to 0} \frac{-\oint_{\partial V} \vec{A} \times d\vec{S}}{\int_V dV}, \qquad \operatorname{grad} \phi = \lim_{r \to 0} \frac{\oint_{\partial V} \phi \, d\vec{S}}{\int_V dV},$$

Important for volume integral formulations: Finite Volume Time Domain (FVTD)

Integration: Stokes, Gauss

Curl equations: Stokes

$$\int_{\partial S} \vec{E}(\vec{r},t)\,\mathrm{d}\vec{\ell} = -\int_{S} \frac{\partial}{\partial t}\vec{B}(\vec{r},t)\,\mathrm{d}\vec{S}$$

$$\int_{\partial S} \vec{H}(\vec{r},t)\,\mathrm{d}\vec{\ell} = \int_{S} \left(\vec{j}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t) \right) \mathrm{d}\vec{S}$$

Div equations: Gauss

$$\int_{\partial V} \vec{D}(\vec{r},t)\,\mathrm{d}\vec{S} = -\int_{V} \rho(\vec{r},t)\,\mathrm{d}V$$

$$\int_{\partial V} \vec{B}(\vec{r},t)\,\mathrm{d}\vec{S} = 0$$

Time-harmonic fields: time separation

$$F(\vec{r}, t) = \Re\left(F(\vec{r}) \cdot e^{-i\omega t}\right)$$

Note: +*j* instead of –*i* in engineering

Notes: Complex fields *F*; time derivative leads to multiplication by $-i\omega$

Fourier Integral (slightly different definitions also used)

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{+i\omega t}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega)\, e^{+i\omega t}$$

Maxwell in Frequency domain

$$\text{curl}\, \vec{E}(\vec{r}) = i\omega\, \vec{B}(\vec{r})$$

$$\text{curl}\, \vec{H}(\vec{r}) = \vec{j}(\vec{r}) - i\omega\, \vec{D}(\vec{r})$$

$$\text{div}\, \vec{D}(\vec{r}) = \rho(\vec{r})$$

$$\text{div}\, \vec{B}(\vec{r}) = 0$$

2D – cylindrical – z-separation – guided waves

$$F(\vec{r}_T, z, t) = \Re\left(F(\vec{r}_T) \cdot e^{i(\gamma z - \omega t)}\right)$$

z derivative leads to multiplication by *iz*

Further simplification of Maxwell equations without curl operator possible!

Static fields: time derivative = zero, frequency = zero

Note: Real fields

Maxwell in statics comes in three parts: electroststics, magnetostatics, currents
Used for the computation of C, L, G (=1/R) respectively

$$\mathrm{curl}\,\vec{E}(\vec{r}) = 0 \qquad \mathrm{curl}\,\vec{H}(\vec{r}) = \vec{j}(\vec{r}) \qquad \mathrm{curl}\,\vec{E}(\vec{r}) = 0$$

$$\mathrm{div}\,\vec{D}(\vec{r}) = \rho(\vec{r}) \qquad \mathrm{div}\,\vec{B}(\vec{r}) = 0 \qquad \mathrm{div}\,\vec{j}(\vec{r}) = 0$$

Simple material equations and decoupling

Linear, homogeneous, isotropic materials: $\vec{D} = \varepsilon \vec{E}$ $\vec{B} = \mu \vec{H}$ $\vec{j}_\sigma = \sigma \vec{E}$

Note: Link to capacitor C, inductor L, conductor G (resistor R=1/G)

Decoupling of 1st order PDEs: get second order decoupled PDEs

Second order spacial derivatives:

$\mathrm{curl\,grad}\,\phi = 0\,,\ \mathrm{div\,curl}\,\vec{A} = 0\,.$ Zero identities used for defining potentials!

Non-zero second order derivatives: scalar and vectorial Laplacian

$$\Delta\phi = \mathrm{div\,grad}\,\phi\,,\quad \Delta\vec{A} = \mathrm{grad\,div}\,\vec{A} - \mathrm{curl\,curl}\,\vec{A}\,.\quad \vec{e}\cdot(\Delta\vec{A}) = \Delta(\vec{e}\cdot\vec{A})\,.$$

Helpful guidelines from «arrow representation»!

Helmholtz and wave equations

Helmholtz homogeneous (k: wave number): $\nabla^2 A + k^2 A = 0$
Inhomogeneous: replace 0 on right hand side
Wave equation: replace k by time derivative (-1/c*...)
Vector: replace A by vector

Wikipedia:
The Helmholtz equation often arises in the study of physical problems involving partial differential equations (PDEs) in both space and time. The Helmholtz equation, which represents the **time-independent** form of the original equation, results from applying the technique of separation of variables to reduce the complexity of the analysis.

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - {c_0}^2 \cdot \nabla^2 \mathbf{E} = 0$$
$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - {c_0}^2 \cdot \nabla^2 \mathbf{B} = 0$$

Electromagnetic wave equations from Wikipedia

Under which conditions do these equations hold?
How many of the field components are independent?

Laplace’s and Poisson’s equations

Wikipedia:
In mathematics, **Laplace's equation** is a second-order partial differential equation named after Pierre-Simon Laplace who first studied its properties.
Laplace's equation and Poisson's equation are the simplest examples of elliptic partial differential equations. Solutions of Laplace's equation are called harmonic functions.

The general theory of solutions to Laplace's equation is known as potential theory. The solutions of Laplace's equation are the harmonic functions, which are important in many fields of science, notably the fields of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials. In the study of heat conduction, the Laplace equation is the steady-state heat equation.

Laplace’s equation is homogeneos, Poisson’s equation is inhomogeneous
Hint: General solution of inh.eq.=general sol.of hom.eq. + special sol.of inh.eq.

Two analytic techniques for solving decoupled DE

Separation of variables for decoupled, homogeneous differential equations:
Select coordinate system; product Ansatz: F(x,y,z,z)=X(x)Y(y)Z(z)T(t)
Hope: ordinary differential equations for X,Y,Z,T obtained and can be solved;
Insert solutions in Maxwell equations! Completeness?

Green's functions for special solutions of decoupled, inhomogeneous DE
Replace inhomogeneity by point singularity in space (Dirac function):

$$L\,G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,') \quad \text{instead of} \quad L\,f(\vec{r}) = g(\vec{r})$$

Solve for all possible locations of the singularity in space and integrate:

$$f(\vec{r}) = \int G(\vec{r},\vec{r}\,')\,g(\vec{r}\,')\,\mathrm{d}V'$$

This special solution often has a physical meaning, e.g. a point charge!
Examples: Coulomb and Ampère integrals have this form!

Why is it easier to find a special solution for a point-like inhomogeneity?
Answer: high symmetry of the configuration!

Derived quantities and important physical laws

- Forces (electric and magnetic): charge or current times E or B
- Voltage (difference of electric potential)
- Current (flux of electric charges)
- Energy (electric and magnetic)
- Poynting vector: E times H
- Power flux (Poynting vector flux)

Conservation laws: charge and energy (time variation in valume – flux!)
symmetries (space, time, charge iversion) - reciprocity