Electromagnetics

Christian Hafner, hafner@ethz.ch

Overview

Traditional areas

Electrostatics
- Capacitors
- High voltage

Magnetostatics
- Inductors
- Motors / Generators

Quasistatics
- Sensors
- Electronic circuits
- Transmission lines

Electrodynamics
- Antennas
- Wave propagation / scattering
- Gratings
- Guided waves
- Resonators...

«Modern areas»
- Integrated optics
- Optical computers
- Nearfield optics
- Nano optics
- Photonic crystals
- Metamaterials
- Plasmonics

Mixed with other disciplines
- Semiconductors
- Lasers, LED, LCD
- Solar cells, Photovoltaics
- RF-MEMS
- (Nano-) Robots
- Bio/medical...

Maxwell equations

$$\mathrm{curl}\,\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$$

$$\mathrm{curl}\,\vec{H}(\vec{r},t) = \vec{j}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t)$$

$$\mathrm{div}\,\vec{D}(\vec{r},t) = \rho(\vec{r},t)$$

$$\mathrm{div}\,\vec{B}(\vec{r},t) = 0$$

Four coupled differential equations
More coupling: description of materials (permeability, permittivity, conductivity, etc.
First order derivatives in space (curl, div) and time

What do the various fields mean?
How can we obtain solutions?

Numerical method: FDTD (Finite Differences, Time Domain)

Definitions of differential operators in 3D space

$$\vec{e}_\ell \cdot \text{grad}\ \phi = \lim_{r\to 0} \frac{\oint_{\partial\ell} \phi\, \mathrm{d}P}{\int_\ell \mathrm{d}\ell}, \quad \vec{e}_n \cdot \text{curl}\ \vec{A} = \lim_{r\to 0} \frac{\oint_{\partial S} \vec{A}\, \mathrm{d}\vec{\ell}}{\int_S \mathrm{d}S}, \quad \text{div}\ \vec{B} = \lim_{r\to 0} \frac{\oint_{\partial V} \vec{B}\, \mathrm{d}\vec{S}}{\int_V \mathrm{d}V},$$

Background: Voltage between 2 points, flux through area, content of a volume

$$U = \int_\ell \vec{v}\ \mathrm{d}\vec{\ell} = \oint_{\partial\ell} \phi\ \mathrm{dP}, \quad \Phi = \int_S \vec{a}\ \mathrm{d}\vec{S} = \oint_{\partial S} \vec{A}\ \mathrm{d}\vec{\ell}, \quad I = \int_V p\ \mathrm{dV} = \oint_{\partial V} \vec{B}\ \mathrm{d}\vec{S}.$$

... law, Stokes law, Gauss law

$$U = \int_\ell \text{grad}\ \phi\ \mathrm{d}\vec{\ell} = \oint_{\partial\ell} \phi\ \mathrm{d}P, \quad \Phi = \int_S \text{curl}\ \vec{A}\ \mathrm{d}\vec{S} = \oint_{\partial S} \vec{A}\ \mathrm{d}\vec{\ell}, \quad I = \int_V \text{div}\ \vec{B}\ \mathrm{dV} = \oint_{\partial V} \vec{B}\ \mathrm{d}\vec{S}.$$

Orthogonal coordinates *u,v,w*, metric coefficients *g*

$$\text{grad}\ \phi = \frac{\phi_{,u}}{g_u}\vec{e}_u + \frac{\phi_{,v}}{g_v}\vec{e}_v + \frac{\phi_{,w}}{g_w}\vec{e}_w. \qquad \text{div}\ \vec{B} = \frac{(g_v g_w B_u)_{,u} + (g_w g_u B_v)_{,v} + (g_u g_v B_w)_{,w}}{g_u g_v g_w},$$

$$\text{curl}\ \vec{B} = \frac{(g_w A_w)_{,v} - (g_v A_v)_{,w}}{g_v g_w}\vec{e}_u + \frac{(g_u A_u)_{,w} - (g_w A_w)_{,u}}{g_w g_u}\vec{e}_v + \frac{(g_v A_v)_{,u} - (g_u A_u)_{,v}}{g_u g_v}\vec{e}_w,$$

Differential operators in 3D space: Volume integrals

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Alternative:

$$\operatorname{div}\vec{B} = \lim_{r\to 0}\frac{\oint\limits_{\partial V}\vec{B}\,\mathrm{d}\vec{S}}{\int\limits_{V}\mathrm{d}V}, \qquad \operatorname{curl}\vec{A} = \lim_{r\to 0}\frac{-\oint\limits_{\partial V}\vec{A}\times\mathrm{d}\vec{S}}{\int\limits_{V}\mathrm{d}V}, \qquad \operatorname{grad}\phi = \lim_{r\to 0}\frac{\oint\limits_{\partial V}\phi\,\mathrm{d}\vec{S}}{\int\limits_{V}\mathrm{d}V},$$

Important for volume integral formulations: Finite Volume Time Domain (FVTD)

Integral form of Maxwell's equations

Curl equations: Integration using Stokes' theorem

$$\int_{\partial S} \vec{E}(\vec{r},t)\,\mathrm{d}\vec{\ell} = -\int_{S} \frac{\partial}{\partial t}\vec{B}(\vec{r},t)\,\mathrm{d}\vec{S}$$

$$\int_{\partial S} \vec{H}(\vec{r},t)\,\mathrm{d}\vec{\ell} = \int_{S} \left(\vec{j}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t) \right) \mathrm{d}\vec{S}$$

Div equations: Integration using Gauss' theorem

$$\int_{\partial V} \vec{D}(\vec{r},t)\,\mathrm{d}\vec{S} = -\int_{V} \rho(\vec{r},t)\,\mathrm{d}V$$

$$\int_{\partial V} \vec{B}(\vec{r},t)\,\mathrm{d}\vec{S} = 0$$

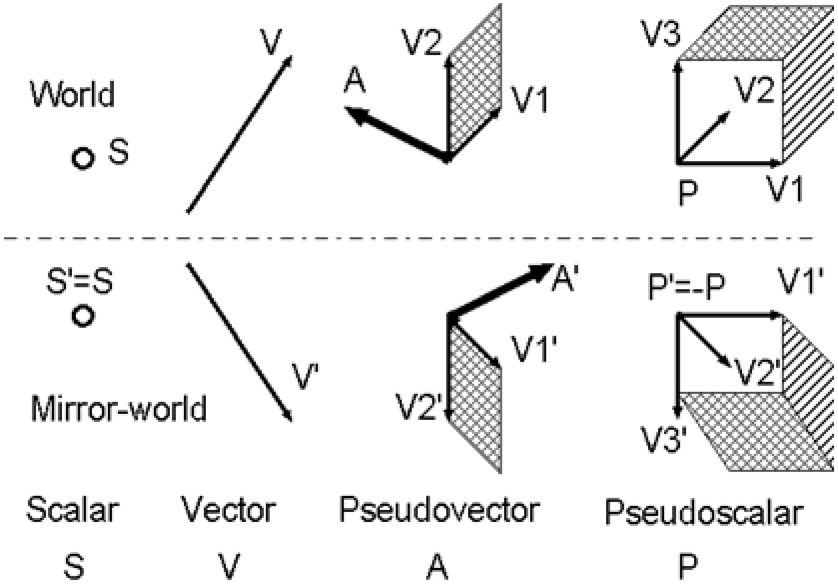
Numerical method: Finite Integral Technique (FIT)

Primitive geometrical objects and Fields

This and much of the following is not only for Maxwell-electromagnetics!

3D space:

0D:point-scalar, 1D:line-vector, 2D:area,pseudo vector, 3D:volume-pseudo scalar



«Mirror world» separtes objects from their pseudo object counterparts.

Adding two scalars or two vectors may make sense, but adding scalar and pseudo scalar or vector and pseudo vecter is as much nonsense as adding a scalar and a vector.

Pseudo objects are characterized by the «missing dimensions»

Vector product: 2 vectors span an area, 1D × 1D → 2D «pseudo: 3D-2D=1D»

Triple product: 1D· (1D × 1D) → 3D «pseudo: 3D-3D=0D»

Differential operators: Link fields with dimension ± 1

+1: «derivatives to the right»

$$s \xrightarrow{grad} \vec{v} \xrightarrow{curl} \vec{a} \xrightarrow{div} p$$

-1: «derivatives to the left»

$$s \xleftarrow{div} \vec{v} \xleftarrow{curl} \vec{a} \xleftarrow{grad} p$$

Derivative of a derivative «in the same direction» is zero!
2nd order non-zero derivatives: Laplacian operators. Note: $\vec{e}\cdot(\Delta\vec{A}) = \Delta(\vec{e}\cdot\vec{A})$.

$$\Delta\phi = \text{div grad}\ \phi\,,$$

$$\Delta\vec{A} = \text{grad div}\ \vec{A} - \text{curl curl}\ \vec{A}\,.$$

Differential operators: Introduction of potentials

When a 1st order (spacial) differential equation is homogenous and the dimension of the field is not 0D or 3D (in 3D space), i.e., the field is not scalar or pseudo scalar:
A potential may be itroduced, such that the homogeneous differential equation is automatically fulfilled. Then, the given field is a (spacial) derivative of the potential.

A field is uniquely defined, when its derivatives «to the left» and «to the right» are known.

A potential is some sort of auxiliary field that is not uniquely defined.
To a scalar potential, one always may add an arbitrary constant (scaling).
For a vector potential, the derivative «in the opposite direction» can be defined arbitrarily (gauge).

Usually, one can derive a 2nd order differential equation for the potential (or for the field itself. This equation contains the Laplacian: Laplace, Poisson, Helmholtz, diffusion, wave equations.

Differential operators: Maxwell schematically

$$\vec{E} \xrightarrow{\text{curl}} -\frac{\partial}{\partial t}\vec{B} \xrightarrow{\text{div}} 0$$

$$0 \xleftarrow{\text{div}} \vec{j} + \frac{\partial}{\partial t}\vec{D} \xleftarrow{\text{curl}} \vec{H}$$

$$\rho \xleftarrow{\text{div}} \vec{D}$$

$$\vec{A} \xrightarrow{\text{curl}} \vec{B} \xrightarrow{\text{div}} 0$$

Notes:

- Maxwell’s equations have no solutions! For obtaining solutions, additional equations are needed: Material equations link fields and provide these equations.
- The electric charge is assumed to be a scalar field here, but one might also assume that it is a pseudo scalar. This would mirror the scheme above.
- There is a formally much simpler 4D notation of Maxwell’s equations, which only contains a homogeneous (→ 4D potential) and an inhomogeneous equation.
- The vector potential A is already intorduced here.

Time-harmonic fields: time separation

$$F(\vec{r},t) = \Re(F(\vec{r}) \cdot e^{-i\omega t})$$

Note: -*i* in physics (measurements)
+*j* instead of –*i* in engineering books +Comsol

Notes: Complex fields *F*; time derivative leads to multiplication by $-i\omega$

Fourier Integral (slightly different definitions also used)

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega t} d\omega \qquad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{+i\omega t} dt$$

Maxwell in frequency domain – used in many numerical methods!

$$\operatorname{curl} \vec{E}(\vec{r}) = i\omega \vec{B}(\vec{r}) \qquad \operatorname{curl} \vec{H}(\vec{r}) = \vec{j}(\vec{r}) - i\omega \vec{D}(\vec{r})$$

$$\operatorname{div} \vec{D}(\vec{r}) = \rho(\vec{r}) \qquad \operatorname{div} \vec{B}(\vec{r}) = 0$$

2D – cylindrical – z-separation – guided waves

$$F(\vec{r}_T, z, t) = \Re\left(F(\vec{r}_T) \cdot e^{i(\gamma z - \omega t)}\right)$$

z derivative leads to multiplication by $i\gamma$

Further simplification of Maxwell equations without curl operator possible!

Note: various other notations possible (if you read different books / papers), e.g.,

$$e^{\gamma z + j\omega t},\ e^{-\gamma z + j\omega t},\ e^{j(\gamma z + \omega t)},\ e^{j(\gamma z - \omega t)}, \ldots$$

Static fields: time derivative = zero, frequency = zero

Note: Real valued fields

Maxwell in statics comes in three parts: electrostatics, magnetostatics, static currents. Used for the computation of C, L, G (=1/R) respectively

$$\operatorname{curl} \vec{E}(\vec{r}) = 0 \qquad \operatorname{curl} \vec{H}(\vec{r}) = \vec{j}(\vec{r}) \qquad \operatorname{curl} \vec{E}(\vec{r}) = 0$$

$$\operatorname{div} \vec{D}(\vec{r}) = \rho(\vec{r}) \qquad \operatorname{div} \vec{B}(\vec{r}) = 0 \qquad \operatorname{div} \vec{j}(\vec{r}) = 0$$

Notes (see also following slides):

Material properties provide links between E,D; H,B; E,j → only 1 unknown vector field for each of the three cases.

Introduce potentials to «solve» a homogeneous equation → insert the potential in the other equation for obtaining a single second order differential equation → try to obtain Laplace or Poisson equations → formal integration (Green's functions).

Simple material equations and decoupling

Linear, homogeneous, isotropic materials: $\vec{D} = \varepsilon \vec{E}$ $\vec{B} = \mu \vec{H}$ $\vec{j}_\sigma = \sigma \vec{E}$

Note: Link to capacitor C, inductor L, conductor G (resistor R=1/G) (see also later)

Decoupling of 1st order PDEs: get second order decoupled PDEs

Second order spacial derivatives:

$\text{curl grad } \phi = 0 \,, \ \text{div curl } \vec{A} = 0 \,.$ Zero identities used for defining potentials!

Non-zero second order derivatives: scalar and vectorial Laplacian

$$\Delta\phi = \text{div grad } \phi \,, \quad \Delta\vec{A} = \text{grad div } \vec{A} - \text{curl curl } \vec{A} \,. \quad \vec{e} \cdot (\Delta\vec{A}) = \Delta(\vec{e} \cdot \vec{A}) \,.$$

Helpful guidelines are obtained from «arrow representation»!

Helmholtz and wave equations

Helmholtz homogeneous (k: wave number): $\nabla^2 A + k^2 A = 0$
Inhomogeneous: replace 0 on right hand side
Wave equation: replace k by time derivative (-1/c*...)
Vector: replace A by vector

Wikipedia:
The Helmholtz equation often arises in the study of physical problems involving partial differential equations (PDEs) in both space and time. The Helmholtz equation, which represents the **time-independent** form of the original equation, results from applying the technique of separation of variables to reduce the complexity of the analysis.

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - {c_0}^2 \cdot \nabla^2 \mathbf{E} \ = \ 0$$
$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - {c_0}^2 \cdot \nabla^2 \mathbf{B} \ = \ 0$$

Electromagnetic wave equations from Wikipedia

Under which conditions do these equations hold?
How many of the field components are independent?

Laplace’s and Poisson’s equations

Wikipedia:
In mathematics, **Laplace's equation** is a second-order partial differential equation named after Pierre-Simon Laplace who first studied its properties.
Laplace's equation and Poisson's equation are the simplest examples of elliptic partial differential equations. Solutions of Laplace's equation are called harmonic functions.

The general theory of solutions to Laplace's equation is known as potential theory. The solutions of Laplace's equation are the harmonic functions, which are important in many fields of science, notably the fields of electromagnetism, astronomy, and fluid dynamics, because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials. In the study of heat conduction, the Laplace equation is the steady-state heat equation.

Laplace’s equation is homogeneos, Poisson’s equation is inhomogeneous
Hint: General solution of inh.eq.=general sol.of hom.eq. + special sol.of inh.eq.

Two analytic techniques for solving decoupled DE

Separation of variables for decoupled, homogeneous differential equations:
Select coordinate system; product Ansatz: F(x,y,z,z)=X(x)Y(y)Z(z)T(t)
Hope: ordinary differential equations for X,Y,Z,T obtained and can be solved;
Insert solutions in Maxwell equations! Completeness?

Green's functions for special solutions of decoupled, inhomogeneous DE
Replace inhomogeneity by point singularity in space (Dirac function):

$$L\,G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,') \quad \text{instead of} \quad L\,f(\vec{r}) = g(\vec{r})$$

Solve for all possible locations of the singularity in space and integrate:

$$f(\vec{r}) = \int G(\vec{r},\vec{r}\,')\,g(\vec{r}\,')\,\mathrm{d}V'$$

This special solution often has a physical meaning, e.g. a point charge!
Examples: Coulomb and Ampère integrals have this form!

Why is it easier to find a special solution for a point-like inhomogeneity?
Answer: high symmetry of the configuration!

Derived quantities and important physical laws

- Forces (electric and magnetic): charge or current times E or B
- Voltage (difference of electric potential, line integral of E field)
- Current (flux of electric charges, closed line integral of H field)
- Energy density (electric ED and magnetic HB)
- Poynting vector: E times H
- Power flux (Poynting vector flux)

Conservation laws: charge and energy (time variation in volume – flux!)
symmetries (space, time, charge iversion) - reciprocity

Link to electronic circuits –primitive elements R,L,C

- Electrostatics $\varepsilon \rightarrow E,D \rightarrow U,Q \rightarrow$ Capacitor $C=Q/U \rightarrow i(t) = C\, du/dt$
- Magnetostatics $\mu \rightarrow H,B \rightarrow I,\Psi \rightarrow$ Inductor $L= \Psi/I \rightarrow u(t) = L\, di/dt$
- Static currents $\sigma \rightarrow E,j \rightarrow U,I \rightarrow$ Resistor $R=U/I \rightarrow u(t) = R\, i$

Notes:

- R,L,C are passive elements connected by 2 wires each
- R converts electric to thermal energy (heater, light bulb) → multiphysics
- L,C store electric/magnetic energy → storage, memory
- L,C static but provide link to time dependence! → filters, resonators, ...
- Derivative/integration of u,i with L,C → analog computers
- Simple computation of R,L,C if linear materials: $D=\varepsilon E$, $B=\mu H$, $j=\sigma E$
- Analytic solutions only if simple geometry (high symmetry)
- Active elements with two wires: current and voltage sources require some energy conversion process (batteries, power generators...) → multiphysics
- More complicated elements with 4 wires: transformers, gyrators
- Semiconductor elements, electron tubes etc. with ≥ 2 wires: links to other disciplines of physics → multiphysics