4. Electrodynamics and Maxwell's Equations

4.1. Introduction

Up to now we have discussed only static fields. When the field sources (electric charge) start to accelerate, the situation becomes different and new effects begin to appear. In this chapter we will analyze the field that is produced by an electric charge in non-stationary movement, i.e. we will analyze dynamic electromagnetic fields.

4.2. Faraday's Law[1]

In previous chapters we have seen the fundamental governing laws of static fields (Ampère's and Gauss's Law) while describing the properties of the electrostatic and magnetostatic fields. The mutual interaction between the electric and magnetic field was not described by either of the mentioned laws. In this section this interaction will be shown in the form of Faraday's law:

$$\oint_{(\Gamma)} \vec{E}(\vec{r}) \cdot d\vec{l}(\vec{r}) = -\frac{\partial \Phi_{S(\Gamma)}}{\partial t} \tag{4.6}$$

It states that the induced voltage in a closed loop (the line integral of electric field over the loop) is equal to the time derivative of the magnetic flux penetrating the surface of the loop, as shown in the Figure 4.2.

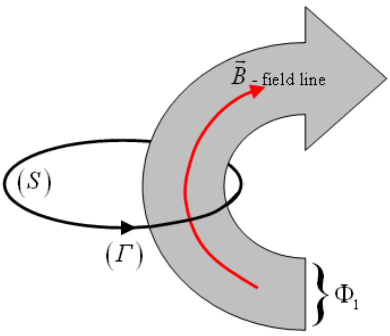

Figure 4.2. Faraday's law; Γ - closed integration curve; S - is the surface bordered by Γ; The red solid line is the field line[2] of vector $\vec{B}$; The gray tube is the magnetic flux Φ_1 penetrating through the surface S.

The magnetic flux through the surface (S) can be calculated as follows:

$$\Phi_S(t) = \iint_{(S)} \vec{B}(\vec{r},t) \cdot \vec{n}(\vec{r})\, dS(\vec{r}) \tag{4.7}$$

Having equation (4.7), Faraday's Law, equation (4.6), can be written in a more appropriate form:

[1] Michael Faraday (1791-1867), British experimental physicist.
[2] The field line is the line tangent to the corresponding field at any point.

$$\oint_{(\Gamma)} \vec{E}(\vec{r}) \cdot d\vec{l}(\vec{r}) = -\frac{\partial}{\partial t} \iint_{(S)} \vec{B}(\vec{r},t) \cdot \vec{n}(\vec{r})\, dS(\vec{r}) \tag{4.8}$$

Since the time derivative has nothing to do with the surface integration in the right-hand side term, the order of operators can be exchanged, i.e. equation (4.8) becomes:

$$\oint_{(\Gamma)} \vec{E}(\vec{r}) \cdot d\vec{l}(\vec{r}) = -\iint_{(S)} \frac{\partial \vec{B}}{\partial t}(\vec{r},t) \cdot \vec{n}(\vec{r})\, dS(\vec{r}) \tag{4.9}$$

or using the already mentioned Stoke's theorem:

$$\iint_{(S)} \nabla \times \vec{E}(\vec{r},t) \cdot \vec{n}(\vec{r})\, dS(\vec{r}) = -\iint_{(S)} \frac{\partial \vec{B}}{\partial t}(\vec{r},t) \cdot \vec{n}(\vec{r})\, dS(\vec{r}) \tag{4.10}$$

Since equation (4.10) holds for an arbitrary surface (S), the vector functions under the surface integrals must be equal, i.e. the differential form of Faraday's law in a vacuum has the form:

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}}{\partial t}(\vec{r},t) \tag{4.11}$$

which is the famous 4^{th} Maxwell's equation valid for the situation when bodies, surfaces and curves of the domain do not move relative to each other.

4.3. Maxwell's[3] Equations in Vacuum

In the previous review of static and dynamic electromagnetic fields the fundamental laws of electromagnetic field theory were presented and explained in detail. All of them but Faraday's law were obtained from steady-state experimental observations. Although it is counterintuitive, these equations remain unchanged by switching from static to time-dependent fields. Let us write once more, for the sake of completeness, all these laws in differential form:

$$\nabla \cdot \vec{E}(\vec{r},t) = \frac{\rho(\vec{r},t)}{\varepsilon_0} \tag{4.12}$$

$$\nabla \times \vec{B}(\vec{r},t) = \mu_0 \vec{J}(\vec{r},t) \tag{4.13}$$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}}{\partial t}(\vec{r},t) \tag{4.14}$$

$$\nabla \cdot \vec{B}(\vec{r},t) = 0 \tag{4.15}$$

Although these equations, as they are written, look simple and mathematically elegant they are not completely correct, i.e. there is one large mathematical problem hidden in them. It was Maxwell who first realised the inconsistency of these equations [1] and modified them into a consistent set of equations that still bear his name. Maxwell published the theoretical results concerning this subject in the year 1862 in his famous paper "On Physical Lines of Force" [1].

[3] James Clerk Maxwell (1831-1879), British theoretical physicist.

Let us analyze, for the moment, equations (4.12-4.15) in order to recognize the previously mentioned inconsistency. Equation (4.14) Faraday's Law, says that a time-dependent magnetic field "induces" an electric field. If this field conversion is possible from magnetic to electric it is logical to expect the existence of a field conversion in the opposite direction. Namely, why not expect that a time-dependent electric field should be able to induce a magnetic field as well? But according to equations (4.12-4.15) this other way around is not possible. This is the signal that something might be wrong with them. At that time Maxwell was of a similar opinion and he wanted to find this missing key that would eventually make these equations complete.
Our analysis will follow Maxwell's idea, i.e. it will start with the equation:

$$\nabla \times \vec{B}(\vec{r},t) = \mu_0 \vec{J}(\vec{r},t) \tag{4.16}$$

By applying the divergence operator to both sides of (4.16), it becomes:

$$0 \equiv \nabla \cdot \left[\nabla \times \vec{B}(\vec{r},t) \right] = \mu_0 \nabla \cdot \vec{J}(\vec{r},t) \overset{(4.5)}{=} -\mu_0 \frac{\partial \rho}{\partial t}(\vec{r},t) \tag{4.17}$$

On the left-hand side we have the identity set to zero because the divergence of the curl is always zero. On the right hand side we have, according to the continuity equation, (4.5), something that is in general not equal to zero. To overcome this mathematical difficulty Maxwell had introduced an additional vector that we, for the moment, will refer to as the "missing term":

$$\nabla \times \vec{B}(\vec{r},t) = \mu_0 \vec{J}(\vec{r},t) + \vec{X}(\vec{r},t) \tag{4.18}$$

By applying the divergence operator to equation (4.18), it becomes:

$$0 = \mu_0 \nabla \cdot \vec{J}(\vec{r},t) + \nabla \cdot \vec{X}(\vec{r},t) \overset{(4.5)}{=} -\mu_0 \frac{\partial \rho}{\partial t}(\vec{r},t) + \nabla \cdot \vec{X}(\vec{r},t) \tag{4.19}$$

From equation (4.19) we have obtained the divergence of the unknown "missing term":

$$\nabla \cdot \vec{X}(\vec{r},t) = \mu_0 \frac{\partial \rho}{\partial t}(\vec{r},t) \tag{4.20}$$

In order to get the "missing term" itself we have to introduce equation (4.12) into equation (4.20):

$$\nabla \cdot \vec{X}(\vec{r},t) = \mu_0 \frac{\partial}{\partial t}\left[\varepsilon_0 \nabla \cdot \vec{E}(\vec{r},t) \right] = \nabla \cdot \left[\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}(\vec{r},t) \right] \tag{4.21}$$

From equation (4.21), the following results:

$$\vec{X}(\vec{r},t) = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}(\vec{r},t) + \vec{f}(t) \tag{4.22}$$

where $\vec{f}(t)$ is an arbitrary vector function depending only on time and the simplest choice, from a practical viewpoint, would be $\vec{f}(t) = 0$. Hence the "missing term" has been discovered as:

$$\vec{X}(\vec{r},t) = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}(\vec{r},t) \tag{4.23}$$

Maxwell called this missing term the *displacement current* and it plays one of the most important roles in the development of electromagnetic theory. As a conclusion, it is useful to write the complete set of Maxwell's equations for a vacuum:

$$\nabla \cdot \vec{E}(\vec{r},t) = \frac{\rho(\vec{r},t)}{\varepsilon_0} \tag{4.24}$$

$$\nabla \times \vec{B}(\vec{r},t) = \mu_0 \vec{J}(\vec{r},t) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}(\vec{r},t) \tag{4.25}$$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}}{\partial t}(\vec{r},t) \tag{4.26}$$

$$\nabla \cdot \vec{B}(\vec{r},t) = 0 \tag{4.27}$$

This additional term that Maxwell referred to as the displacement current produced a completely new effect that was not known before. Namely, it led Maxwell towards the prediction of the existence of *electromagnetic waves* that would be experimentally confirmed much later[4].

4.4. Maxwell's Equations in Macroscopic Media

In the previous chapters on the electrostatic and magnetostatic field we have introduced the vector of electric displacement, $\vec{D}$, and the vector of magnetic field, $\vec{H}$, as a sort of response of the material to the external field. Having these two vectors in addition to the vectors $\vec{E}$ and $\vec{B}$, one can rewrite the Maxwell's equations in general and compact form. Namely, equation (4.24) will be transformed using equations (1.45 and 1.46) into:

$$\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t) \tag{4.28}$$

In a similar way equation (4.25) will be modified using equations (1.59, 1.60, 1.45 and 1.46) into:

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \vec{D}}{\partial t}(\vec{r},t) \tag{4.29}$$

Equations (4.28 and 4.29) along with equations (4.26 and 4.27) form the complete set of Maxwell's equations for macroscopic media, as follows:

$$\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t) \qquad (4.28)$$

[4] Heinrich Hertz (1857-1894), German physicist who experimentally confirmed the existence of electromagnetic waves in the year 1888.

$$\nabla \cdot \vec{B}(\vec{r},t) = 0 \tag{4.27}$$

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial \vec{B}}{\partial t}(\vec{r},t) \tag{4.26}$$

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial \vec{D}}{\partial t}(\vec{r},t) \tag{4.29}$$

With these four very mathematically elegant equations nearly all physical processes related to electromagnetic fields can be described and solved using a certain set of boundary conditions[5]. It is difficult to find in other fields of science such a compact set of equations describing huge amounts of empirical knowledge and theory. This fact shows how important and progressive the work of Maxwell was for the development of modern electromagnetic field theory.

4.5. Interface Conditions Over a Boundary Between Different Media

By using the set of Maxwell's equations given in the previous chapter in differential form and transferring them into integral form, one can draw important conclusions about the field behaviour at the interface between different materials. Let us consider the situation depicted in Figure 4.3.

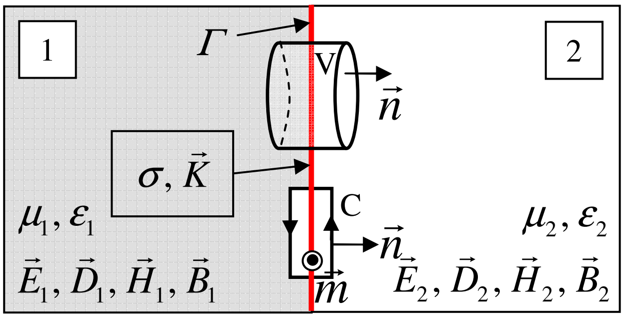

Figure 4.3. Interface between different materials (materials are different in an electromagnetic sense); *V* – small cylindrical volume; *C* – small contour; σ - surface density of electric charge over interface; $\vec{K}$ - surface current density over interface; Γ - surface boundary between two domains.

By performing the surface integral of equation (4.28) over the boundary of the cylindrical volume, V, and by analyzing the limit of this integral when the height of the cylinder tends towards zero one can easily obtain the following [2]:

$$\left(\vec{D}_1 - \vec{D}_2\right) \cdot \vec{n} = \sigma \tag{4.30}$$

By following the same procedure but starting from equation (4.27) the second interface condition [2] is obtained:

$$\left(\vec{B}_1 - \vec{B}_2\right) \cdot \vec{n} = 0 \tag{4.31}$$

[5] The Boundary Condition (BC) is a prescribed value of electromagnetic field over the boundary of the computational domain. The Boundary Conditions restrict an infinite set of possible (partial) solutions of partial differential equations to the unique solution of the concrete problem defined by the geometry and material properties. Mathematicaly speaking, having BCs sets problems of electromagnetic fields computation to become unique (there is only one solution).

If we now integrate equation (4.29) along the closed curve (C) and analyze the limiting process when the width of the loop tends towards zero, we obtain [2]:

$$\left(\vec{H}_2 - \vec{H}_1\right) \times \vec{n}(\vec{r}) = \vec{K} \tag{4.32}$$

An identical approach can be applied to equation (4.26) producing the following interface condition [2]:

$$\left(\vec{E}_2 - \vec{E}_1\right) \times \vec{n}(\vec{r}) = 0 \tag{4.33}$$

Equations (4.30-4.33) are the prominent interface conditions for the electromagnetic field that have to be fulfilled over the boundary between two materials with different electromagnetic properties. As will be shown later, they play a very important role in both analytical and numerical field computations.

4.6. Time-Harmonic Fields

Our equations presented up to now in the chapter on electrodynamics have been written in general for any kind of time-varying field. In the case of harmonic time-variation it is possible to introduce a complex representation of the field vectors which significantly simplifies Maxwell's equations. Namely, the complex representation of the field is defined as follows [2]:

$$\vec{F}\left(\vec{r},t\right) = \mathrm{Re}\left\{\underline{\vec{F}}(\vec{r}) \cdot e^{j\omega t}\right\},\ \underline{\vec{F}}(\vec{r}) \text{ - complex representation of the field vector} \tag{4.34}$$

This transformation allows us to replace the time derivative in Maxwell's equations with the following algebraic term:

$$\frac{\partial}{\partial t}\left(e^{j\omega t}\right) = j\omega \cdot e^{j\omega t} \tag{4.35}$$

By introducing equations (4.34 and 4.35) into the set of Maxwell's equations we obtain the following:

$$\nabla \cdot \underline{\vec{D}}(\vec{r}) = \underline{\rho}(\vec{r}) \tag{4.36}$$

$$\nabla \cdot \underline{\vec{B}}(\vec{r}) = 0 \tag{4.37}$$

$$\nabla \times \underline{\vec{E}}(\vec{r}) = -j\omega \underline{\vec{B}}(\vec{r}) \tag{4.38}$$

$$\nabla \times \underline{\vec{H}}(\vec{r}) = \underline{\vec{J}}(\vec{r}) + j\omega \cdot \underline{\vec{D}}(\vec{r}) \tag{4.39}$$

Equations (4.36-4.39) are Maxwell's equations in the frequency domain. All vectors here are complex representations.

4.7. Wave Equations

In Section 4.3 we mentioned that the major impact of Maxwell's equations was the prediction of electromagnetic waves. The waves were then experimentally confirmed by Heinrich Hertz in the year 1888. In this section, based on Maxwell's equations (4.26-4.29) we will derive the wave equations for both the electric and magnetic field. If we perform the curl operator of equation (4.26) we obtain the following:

$$\nabla\times\nabla\times\vec{E}(\vec{r}) = -\frac{\partial}{\partial t}\nabla\times\vec{B}(\vec{r},t) \qquad (4.40)$$

Equation (4.40) can be further transformed by using the identity $\nabla\times\nabla\times\vec{E} = \nabla\left(\nabla\cdot\vec{E}\right) - \Delta\vec{E}$, the relations $\vec{B} = \mu\vec{H}$, $\vec{D} = \varepsilon\vec{E}$, $\vec{J} = \vec{J}_S + \sigma\vec{E} = \sigma\vec{E}$ (the source current is assumed to be zero, $\vec{J}_S = 0$) and equation (4.39), as follows:

$$\Delta\vec{E}(\vec{r},t) - \mu\sigma\frac{\partial\vec{E}}{\partial t}(\vec{r},t) - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}(\vec{r},t) = 0 \qquad (4.41)$$

Equation (4.41) has been derived with the assumption that: $\rho = 0$, i.e. the charge does not exist in the domain of our interest (free-space). In the same way one can derive the following wave equation for magnetic field:

$$\Delta\vec{H}(\vec{r},t) - \mu\sigma\frac{\partial\vec{H}}{\partial t}(\vec{r},t) - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}(\vec{r},t) = 0 \qquad (4.42)$$

The same equations in the frequency domain (harmonic fields) have the following form:

$$\Delta\underline{\vec{E}}(\vec{r}) + \left(\omega^2\mu\varepsilon - j\omega\mu\sigma\right)\underline{\vec{E}}(\vec{r}) = 0 \qquad (4.43)$$

$$\Delta\underline{\vec{H}}(\vec{r}) + \left(\omega^2\mu\varepsilon - j\omega\mu\sigma\right)\underline{\vec{H}}(\vec{r}) = 0 \qquad (4.44)$$

The wave equations have been written for material that has losses (conductibility is not equal to zero, $\sigma \neq 0$). In the case of an ideal dielectric the equations become a bit simpler:

$$\Delta\underline{\vec{E}}(\vec{r}) + \omega^2\mu\varepsilon\underline{\vec{E}}(\vec{r}) = 0 \qquad (4.45)$$

$$\Delta\underline{\vec{H}}(\vec{r}) + \omega^2\mu\varepsilon\underline{\vec{H}}(\vec{r}) = 0 \qquad (4.46)$$

Equations (4.45) and (4.46) are called the vector Helmholtz's equations. Apparently, the wave equations are independent from each other and the wave propagation problem can be solved by using a single one along with its corresponding boundary conditions. The price that has been paid is the increase of the order of the wave equations (2nd order) compared to the initial set of Maxwell's equations (1st order).

4.8. "Retarded" Vector and Scalar Potentials

In the theory of electrostatic and magnetostatic fields we have seen the definition of scalar electric and vector magnetic potentials. Here, this subject will be revised in terms of electrodynamics and electric-magnetic field coupling. Let us first consider the following couple of Maxwell's equations related to magnetic field:

$$\nabla\cdot\vec{B} = 0,\ \nabla\times\vec{H} = \vec{J}_S + \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad (4.47)$$

Accordingly, the vector $\vec{B}$ can be described as a curl of a certain vector called the vector magnetic potential:

$$\boxed{\vec{B} = \nabla \times \vec{A}} \tag{4.48}$$

We can apply a similar procedure for the remaining Maxwell's equations related to electric field:

$$\nabla \cdot \left(\varepsilon \vec{E}\right) = \rho, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.49}$$

In the present form the set of equations (4.49) can not be directly used for the definition of scalar electric potential because the curl of electric field is not equal to zero. Therefore this should be written in a slightly different form using equation (4.49) and the definition of vector potential, equation (4.48):

$$\nabla \cdot \left(\varepsilon \vec{E}\right) = \rho, \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \tag{4.50}$$

Based on equation (4.50), the scalar electric potential can be introduced as follows:

$$\boxed{\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \cdot \Phi} \tag{4.51}$$

It is now very simple to derive the wave equations of the potentials. By introducing equations (4.48 and 4.51) into the curl equation, (4.47), we obtain the following wave equation for vector magnetic potential:

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A}\right) = \vec{J}_S + \sigma \vec{E} + \varepsilon \frac{\partial}{\partial t}\left(-\nabla \cdot \Phi - \frac{\partial \vec{A}}{\partial t}\right) \tag{4.52}$$

After some simple mathematical manipulations ($\nabla \times \nabla \times \vec{A} = \nabla\left(\nabla \cdot \vec{A}\right) - \Delta \vec{A}$) this turns into the following equation:

$$\nabla\left(\nabla \cdot \vec{A}\right) - \Delta \vec{A} + \sigma\left(\nabla \cdot \Phi + \frac{\partial \vec{A}}{\partial t}\right) + \mu\varepsilon \frac{\partial}{\partial t}\left(\nabla \cdot \Phi + \frac{\partial \vec{A}}{\partial t}\right) = \mu \vec{J}_S \tag{4.53}$$

Equation (4.48), i.e. the prescription of its curl operator is not sufficient to uniquely define the vector magnetic potential. Therefore in addition to this we have to prescribe its divergence operator as well, which is a task with a certain level of freedom. Namely, since we want to obtain the wave equation that contains only the unknown vector magnetic potential, we prescribe the divergence of the vector potential in such a way that equation (4.53) becomes simpler and without scalar electric potential. Apparently, it would be very useful to apply the following gauge [2, 3]:

$$\boxed{\nabla \cdot \vec{A} = -\sigma\Phi - \mu\varepsilon\frac{\partial \Phi}{\partial t}} \tag{4.54}$$

As opposed to Coulomb's gauge that we had in magnetostatics ($\nabla \cdot \vec{A} = 0$) we obtain in electrodynamics the so-called Lorentz's gauge (4.54). Having gauge (4.54), the wave equation of vector potential becomes far simpler:

$$\boxed{\Delta\vec{A} - \sigma\frac{\partial \vec{A}}{\partial t} - \mu\varepsilon\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\vec{J}_S} \tag{4.55}$$

By introducing the definition of scalar electric potential (4.51) into the divergence equation of (4.49) we obtain the wave equation of scalar electric potential:

$$\boxed{\Delta\Phi - \sigma\frac{\partial \Phi}{\partial t} - \mu\varepsilon\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon}} \tag{4.56}$$

In the case of loss-free material equations (4.55 and 4.56) reduce to the following simpler form of wave equations in ideal dielectrics:

$$\Delta\vec{A} - \mu\varepsilon\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\vec{J}_S \tag{4.57}$$

$$\Delta\Phi - \mu\varepsilon\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon} \tag{4.58}$$

In the theory of electrostatic and magnetostatic fields we have seen the solution of Poisson's vector and scalar equations for magnetic vector- and electric scalar- potentials respectively in integral form. The solution of wave equations (4.57 and 4.58) will have a similar form with one important difference. Namely, the potential will be "retarded" due to the finite speed of electromagetic waves. The "retarded potentials" as solutions of equations (4.57 and 4.58) can be written in integral form as follows:

$$\vec{A}(\vec{r},t) = \frac{\mu}{4\pi}\iiint_{(V)} \frac{\vec{J}\left(\vec{r}\,', t - \frac{|\vec{r}\,' - \vec{r}|}{v}\right)}{|\vec{r}\,' - \vec{r}|} dV(\vec{r}\,') \tag{4.59}$$

$$\Phi(\vec{r},t) = \frac{1}{4\pi\varepsilon}\iiint_{(V)} \frac{\rho\left(\vec{r}\,', t - \frac{|\vec{r}\,' - \vec{r}|}{v}\right)}{|\vec{r}\,' - \vec{r}|} dV(\vec{r}\,') \tag{4.60}$$

It is quite simple to transfer the wave equations and integral solutions from the given time-domain into frequency-domain. This is strongly recommended to the students for exercise. It is also worth mentioning that the potentials can be defined in some other form as well. For example there are such potentials as the vector electric potential, the scalar magnetic potential etc. Each of these particular definitions is useful for certain specific purposes. The potentials that we have presented here have more general meaning and they are very helpful either to reduce the problem from vector to scalar field computation (electric potential) or to ensure an appropriate integral form for field computation and to reduce the level of difficulty when imposing boundary conditions (vector magnetic potential). Therefore it is important to understand the presented theoretical background of electric and magnetic potential.

4.9. Poynting Theorem

In this section one of the most important theorems in electromagnetics will be presented. The Poynting[6] theorem states the energy conservation law for an arbitrary domain in an electromagnetic field. Let us consider the domain presented in Figure 4.4.

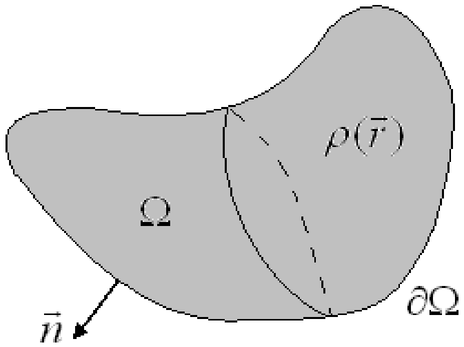

Figure 4.4. A finite domain Ω in an electromagnetic field is presented; This is used for the derivation of the Poynting theorem.

It is assumed that a volume charge density exists in the domain Ω and that this charge moves with a speed $\vec{v}_S$ due to the force produced by the source $\vec{F}_S$. For an infinitely small element of the volume dV, the amount of work carried out by the source can be calculated as:

$$dW_S = \vec{F}_S \cdot \vec{v}_S \, dt \tag{4.61}$$

where dt is the elemental time interval. By assuming that the force has an electric nature the equation (4.61) becomes:

$$dW_S = -\rho \, dV \, \vec{E} \cdot \vec{v}_S \, dt \tag{4.62}$$

where ρ is the volume charge density, dV is the elemental volume and $\vec{E}$ is the electric field existing in the volume. From Section 2 (see equation (2.5)) we know that the volume current density of the source can be calculated as $\vec{J}_S = \rho \, \vec{v}_S$. Thus equation (4.62) reads:

$$dW_S = -\vec{E} \cdot \vec{J}_S \, dV \, dt \tag{4.63}$$

If we divide equation (4.63) by the elemental time dt, we obtain the elemental power of the source released in the elemental volume dV. To compute the total power of the source we have to integrate it over the volume Ω:

$$P_S = -\iiint_{(\Omega)} \vec{E} \cdot \vec{J}_S \, dV \tag{4.64}$$

[6] John Henry Poynting (1852 – 1914), English physicist.

On the other hand, from Maxwell's equations the following statement for the source current density can be obtained:

$$\vec{J}_S = \nabla\times\vec{H} - \vec{J} - \frac{\partial\vec{D}}{\partial t} \tag{4.65}$$

By including equation (4.65) into equation (4.64) we obtain the following:

$$P_S = -\iiint_{(\Omega)} \vec{E}\cdot\left(\nabla\times\vec{H}\right)dV + \iiint_{(\Omega)} \vec{E}\cdot\vec{J}\,dV + \iiint_{(\Omega)} \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\,dV \tag{4.66}$$

The first term on the right-hand side of equation (4.66) can be transferred using a well known vector formula [4]:

$$\nabla\left(\vec{a}\times\vec{b}\right) = \vec{b}\cdot\nabla\times\vec{a} - \vec{a}\cdot\nabla\times\vec{b} \tag{4.67}$$

Thus equation (4.66) becomes:

$$P_S = \iiint_{(\Omega)} \nabla\cdot\left(\vec{E}\times\vec{H}\right)dV - \iiint_{(\Omega)} \vec{H}\cdot\left(\nabla\times\vec{E}\right)dV + \iiint_{(\Omega)} \vec{E}\cdot\vec{J}\,dV + \iiint_{(\Omega)} \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\,dV \tag{4.68}$$

Gauss's theorem [5] can be directly applied to the first term of the right–hand side of equation (4.68). The second term can be modified using Maxwell's equation (4.26). Thus the Poynting theorem is obtained in its well known form:

$$P_S = \oiint_{(\partial\Omega)} \left(\vec{E}\times\vec{H}\right)\cdot d\vec{S} + \iiint_{(\Omega)} \vec{E}\cdot\vec{J}\,dV + \iiint_{(\Omega)} \left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)dV \tag{4.69}$$

The first term on the right hand side represents the radiated electromagnetic energy through the boundary of the domain (oriented outwards). The second term describes the Ohmic losses of the domain itself. And the third term is the contribution of the source to the power of the electromagnetic field inside the domain.

4.10. Plane Waves in Perfect Dielectrics

In this section we will analyze the propagation of a plane wave in an ideal dielectric described by the magnetic permeability μ, the dielectric permittivity ε, and the electric conductivity $\sigma = 0$. For such a domain one can write the following wave equation according to equation (4.41):

$$\Delta\vec{E}(\vec{r},t) - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}(\vec{r},t) = 0 \tag{4.70}$$

Equation (4.70) describes a general electromagnetic wave (EMW) in an ideal dielectric. We will limit our analysis to the plane EMW. Such an EMW has an electric field constant in the plane which is perpendicular to its propagation direction. Let us assume that our coordinate system is defined so that the z-direction is the same as the propagation direction. It is also obvious that we do not loose the generality if we position our system in such a way that the y-component of the electric field is equal to zero. Thus the electric field has the following form:

$$\vec{E}(z,t) = E_x(z,t)\cdot\vec{e}_x + E_z(z,t)\cdot\vec{e}_z \tag{4.71}$$

Furthermore, let us assume that our dielectric is homogenous, linear, isotropic and without free electric charge. Therefore, according to equation (1.37) we can write an additional condition for the electric field:

$$\nabla\cdot\vec{E} = 0 \;\Rightarrow\; \frac{\partial E_z}{\partial z} = 0 \;\Rightarrow\; \text{The most simple solution is: } E_z = 0 \tag{4.72}$$

Thus, with an appropriate coordinate system without loosing the generality we can write the electric field of the plane EMW in the following form:

$$\vec{E}(z,t) = E_x(z,t)\cdot\vec{e}_x \tag{4.73}$$

Having equation (4.73) the wave equation (4.70) becomes:

$$\frac{\partial^2 E_x}{\partial z^2}(z,t) - \mu\varepsilon\frac{\partial^2 E_x}{\partial t^2}(z,t) = 0 \tag{4.74}$$

There is a well known solution of the wave equation (4.74). In general it has the following form [6]:

$$E_x(z,t) = E^+\; f\left(t-\frac{z}{c}\right) + E^-\; g\left(t+\frac{z}{c}\right) \tag{4.75}$$

where c is the speed of the EMW propagation $c = 1\big/\sqrt{\mu\varepsilon}$, f and g are arbitrary functions and E^+, E^- are constants. It is also easy to show that the first term on the right-hand side of equation (4.75) is the wave propagating in the z direction and the second term is the wave propagating in the opposite direction.
The corresponding magnetic field can be derived using the Maxwell's equation (4.26):

$$\nabla\times\vec{E}(\vec{r},t) = -\mu\frac{\partial\vec{H}}{\partial t}(\vec{r},t) \;\Rightarrow\; H_x = 0\,,\; -\mu\frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z}\,,\; H_z = 0 \tag{4.76}$$

If we consider only the progressive wave in (4.75) with an amplitude E^+ and if we introduce it into (4.76) it is easy (if we neglect the time constant term after integration) to obtain the following:

$$H_y(z,t) = \frac{1}{\mu c}\, E^+\; f\left(t-\frac{z}{c}\right) = \sqrt{\frac{\varepsilon}{\mu}}\; E_x(z,t) \tag{4.77}$$

The constant of the proportionality between the magnetic and electric field is the so-called wave impedance of the domain:

$$Z = \sqrt{\frac{\varepsilon}{\mu}} \tag{4.78}$$

This derivation show us several important statements regarding the plane EMW:

1. the magnetic field and electric field are perpendicular to each other at any point in space,
2. both vectors of the electric and magnetic fields are perpendicular to the propagation direction,
3. the Poynting vector ($\vec{E} \times \vec{H}$) (see equation (4.69)) is parallel to the propagation direction, and
4. the ratio between the intensity of the magnetic field and the intensity of the electric field is equal to the wave impedance of the media.

4.11. References

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