Exam. 19 Question 19:

1)
$$\begin{cases} \dfrac{\partial u}{\partial x} + a\dfrac{\partial v}{\partial t} = 0 & \dfrac{\partial}{\partial t} \\ \dfrac{\partial v}{\partial x} + b\dfrac{\partial u}{\partial t} = 0 & \dfrac{\partial}{\partial x} \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial t} + a\dfrac{\partial^2 v}{\partial t^2} = 0 \\ \dfrac{\partial^2 v}{\partial x^2} + b\dfrac{\partial^2 u}{\partial x \partial t} = 0 \end{cases} \Rightarrow \quad \frac{\partial^2 v}{\partial x^2} + b\left(-a\frac{\partial^2 v}{\partial t^2}\right) = 0$$

$$c^2\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} = 0$$

$$\boxed{c = \frac{1}{\sqrt{ab}}}$$

2) $v(x,t) = Re\{V_0 \exp(i(kx - \omega t))\}$

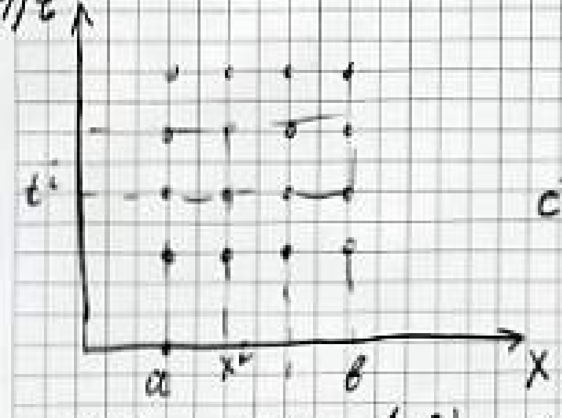
$$c^2(-k^2)v\cancel{(x,t)} - \omega^2 \cancel{v(x,t)} = 0$$

$$\boxed{k^2 = \left(\frac{\omega}{c}\right)^2}$$

3) $u(x,t) = \int a\dfrac{\partial v}{\partial t}dx = a\int Re\{V_0(i\omega)\exp(i(kx-\omega t))\}dx =$

$= a\, Re\{V_0\left(\dfrac{\omega}{k}\right)\exp(i(kx - \omega t))\}$

4)



$$c^2\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} = 0$$

$$c^2 \begin{cases} x_i = a + ih, \quad i = 1, \ldots, m \quad h = \dfrac{b-a}{m} \\ t^k = k\Delta t \quad k = 0, 1, \ldots \end{cases}$$

$$\frac{\partial v}{\partial x} = \frac{v(x + \frac{1}{2}h) - v(x - \frac{1}{2}h)}{h}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\left(\dfrac{v(x+h) - v(x)}{h}\right) - \left(\dfrac{v(x) - v(x-h)}{h}\right)}{h} = \frac{v(x+h) - 2v(x) + v(x-h)}{h}$$

$\cancel{\dfrac{\partial^2 v}{\partial t^2} = v(t\ldots}$

$$\frac{\partial^2 V}{\partial x^2} \simeq \frac{V_{i+1}^{K} - 2V_{i}^{K} + V_{i+1}^{K}}{\Delta x}$$

$$\frac{\partial^2 V}{\partial t^2} \simeq \frac{V_{i}^{K+1} - 2V_{i}^{K} + V_{i}^{K-1}}{\Delta t}$$

$$\boxed{\frac{V_i^{K+1} - 2V_i^{K} + 2V_i^{K-1}}{(\Delta t)^2} = c^2 \frac{V_{i+1}^{K} - 2V_i^{K} + V_{i-1}^{K}}{(\Delta x)^2}}$$

stability $\boxed{\Delta t\, c \leq \Delta x}$

Question 21.

Ex 21.

a)

$$f_{nm} = \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad n = 0, 1, 2, \quad m = 0, 1, 2$$

$a = 0.4 \quad b = 0.1 (m)$

$f_{nm} = \frac{1}{2} ?$

$$\begin{cases} f_{10} = \frac{1}{2\sqrt{\mu\varepsilon}} \boxed{2.5} \simeq 0.37 \,(GHz) & m = 1, n = 0 \\ f_{01} = \frac{1}{2\sqrt{\mu\varepsilon}} \boxed{10} \simeq 1.49 \,(GHz) & n = 1, m = 0 \\ f_{20} = \frac{1}{2\sqrt{\mu\varepsilon}} \boxed{5} \simeq \boxed{0.748} \,(GHz) & m = 2, n = 0 \end{cases}$$

b)

$n_z = 2.5$

$n_x = 1.$

$$K_x^2 + K_z^2 = \left(\frac{\omega}{c}\right)^2 \qquad \boxed{K_y = 0}$$

$$\omega = c\sqrt{K_x^2 + K_y^2} = c\sqrt{\left(\frac{2\pi}{\lambda_x}\right)^2 + \left(\frac{2\pi}{\lambda_z}\right)^2} = c\sqrt{\left(\frac{2\pi}{\left(\frac{0.4}{1}\right)}\right)^2 + \left(\frac{2\pi}{\left(\frac{1.5}{2.5}\right)}\right)^2}$$

$$\boxed{f = \frac{\omega}{2\pi} = \frac{\omega}{2\pi} \simeq 0.9 \text{ GHz}}$$

$f = \frac{1}{T}$

$\omega = \frac{2\pi}{T} = 2\pi f$

Exercise 20:

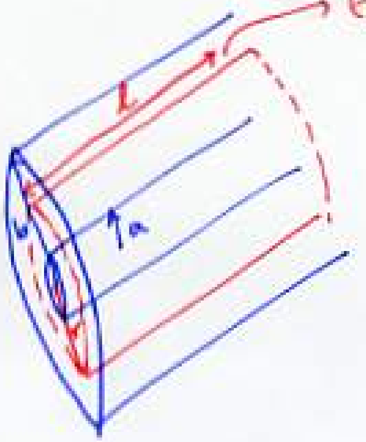
a) Mathematically Gauss' law says: (free-space).

$$\nabla . E = \frac{\rho}{\epsilon_0} \quad \text{or in the integral form:} \quad \oint_S E . dA = \frac{Q}{\epsilon_0}.$$

* Physically, the total E-field (normal component) pointing outside of a closed surface (because dA points out) is equal to the total charge inside the surface devided by the permittivity value.

b)

Gaussian surface.



FOR region 1:

the total charge inside: $\sigma . L$

the total area of the Gaussian surface : $2\pi r . l$

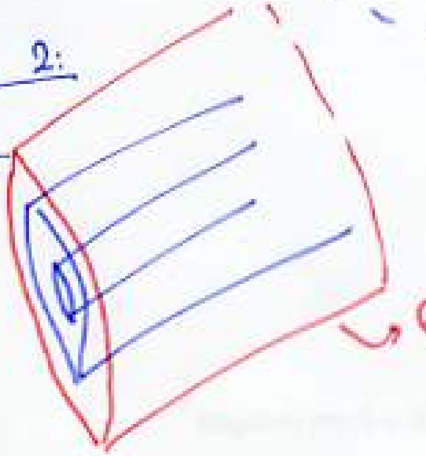
by Gauss' Law:

$$E_r . 2\pi r . \cancel{L} = \frac{\sigma . \cancel{L}}{\epsilon_r . \epsilon_0}$$

$$\Rightarrow \boxed{\vec{E} = \frac{\sigma}{\epsilon_r \epsilon_0 2\pi r} \hat{r}} \text{ in region 1.}$$

FOR region 2:

~~the Gaussian~~



Gaussian surface.

the total charge inside: $\sigma - \sigma = 0$

$$\Rightarrow \boxed{\vec{E} = 0} \text{ in region 2}$$

c) $$\Delta V = \int_a^b E . dl$$

$$= \int_a^b \frac{\sigma}{\epsilon_r \epsilon_0 2\pi r} dr = \boxed{\frac{\sigma}{\epsilon_r \epsilon_0 2\pi} \ln\left(\frac{b}{a}\right)}$$

d) The potential must obey the Laplace eqn. in ~~medium~~ region 1.

$$\nabla V = 0. \qquad V(r=a) = V_0$$

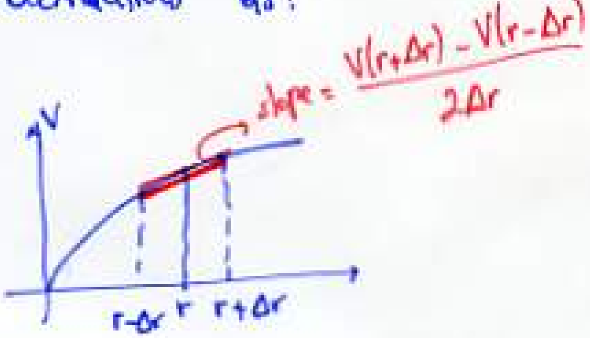
$$V(r=b) = V_0 - \frac{\sigma}{\epsilon_r \epsilon_0 2\pi} \ln\left(\frac{b}{a}\right)$$ (potential increases from negative charge to positive)

Since no dependency on θ or z directions, the laplacian in cylindrical coordinates becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) = 0$$ ⇒ apply a central difference to that equation as we did on class.

Apply the central differences to the derivatives as:

$$\frac{\partial V}{\partial r} = \frac{V(r+\Delta r) - V(r-\Delta r)}{2\Delta r}$$



Exercise: 22

a) a) Forward, b) Backward ~~[illegible]~~ c) d) Central e) central

b) c) 4

c) a) 10

d) d) 4

e) e

f) b) {because of the definition of basis functions}.

g) Poisson's eqn:

$\nabla^2 \phi = f$

* Potential dist. of stationary charges.
* Heat eqn. with source.

Laplace's eqn:

$\nabla^2 \phi = 0.$

* Stationary Heat eqn. with no source. (i.e fixed temp on boundary)
* Potential of an E-field in a source free-region.
* Gravitational pot. eq.