Exercise 5

In free space ($z \leq 0$) a plane wave described by

$$\vec{H}_i = 10\cos(10^8 t - \beta z)\vec{e}_x \, \tfrac{\text{mA}}{\text{m}} \tag{1}$$

is incident normally on a lossless medium ($\varepsilon_r = 2$, $\mu_r = 8$ for $z \geq 0$) as shown in fig. 1. Determine the reflected wave $\vec{H}_r$, $\vec{E}_r$ and the transmitted wave $\vec{H}_t$, $\vec{E}_t$.

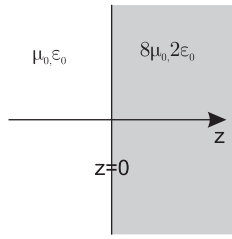

Figure 1: Plane wave incident normally on medium

Solution:

The wavenumber and wave impedance are defined in free space, i.e. medium 1, as

$$\beta_1 = \frac{\omega}{c} = \frac{1}{3} \tag{2}$$

$$Z_1 = Z_0 = 120\pi \tag{3}$$

with $\omega = 10^8 \frac{1}{\text{s}}$. Furthermore they yield for the lossless medium 2 to

$$\beta_2 = \omega\sqrt{\mu\varepsilon} = \frac{4}{3} \tag{4}$$

$$Z_2 = Z_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 2Z_0. \tag{5}$$

Starting with Ampere's law

$$\begin{aligned}
\nabla \times \vec{H}_i &= \varepsilon_1 \frac{d\vec{E}_i}{dt} \\
\vec{e_y}\frac{dH_i}{dz} &= 10\beta_1 \sin(\omega t - \beta z)\vec{e_y} \, \tfrac{\text{mA}}{\text{m}} \\
\vec{E}_i &= 10 \, \tfrac{\text{mA}}{\text{m}} \tfrac{\beta_1}{\varepsilon_1} \vec{e_y} \int \sin(\omega t - \beta_1 z)\, dt \\
\vec{E}_i &= -10 \, \tfrac{\text{mA}}{\text{m}} \tfrac{\beta_1}{\omega\varepsilon_1} \cos(\omega t - \beta_1 z)\vec{e_y}
\end{aligned}$$

and utilising the relation

$$\frac{\beta_1}{\omega\varepsilon_1} = \frac{\omega\sqrt{\mu_1\varepsilon_1}}{\omega\varepsilon_1} = \sqrt{\frac{\mu_1}{\varepsilon_1}} = Z_1$$

the incident electric fields yield to

$$\vec{E}_i = -10Z_1 \cos(10^8 t - \frac{1}{3}z)\vec{e}_y \, \tfrac{\text{mV}}{\text{m}}. \tag{6}$$

Since the wave incident normally on the lossless medium 2, the reflected wave amplitude is defined as

$$\frac{|\vec{E}_r|}{|\vec{E}_i|} = \Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{1}{3}. \tag{7}$$

Thus the reflected wave results in

$$\vec{E}_r = \Gamma\vec{E}_i = -\frac{10}{3}Z_1 \cos(\omega t + \beta_1 z)\vec{e}_y \, \tfrac{\text{mV}}{\text{m}}, \tag{8}$$

$$\vec{H}_r = \Gamma\vec{H}_i = \frac{10}{3}\cos(\omega t + \beta_1 z)\vec{e}_x \, \tfrac{\text{mA}}{\text{m}}. \tag{9}$$

The amplitude of the transmitted wave is defined as

$$\frac{|\vec{E}_r|}{|\vec{E}_i|} = T = 1 + \Gamma = \frac{2Z_2}{Z_2 + Z_1} = \frac{4}{3}. \tag{10}$$

Consequently the transmitted wave yield to

$$\vec{E}_t = T\vec{E}_i = -\frac{40}{3}Z_2 \cos(\omega t - \beta_2 z)\vec{e}_y \, \tfrac{\text{mV}}{\text{m}}, \tag{11}$$

$$\vec{H}_t = T\vec{H}_i = \frac{20}{3}\cos(\omega t - \beta_2 z)\vec{e}_x \, \tfrac{\text{mA}}{\text{m}}. \tag{12}$$

Exercise 6

A parallel plate waveguide, see fig. 2, is illuminated by two plane waves. The superposition of the two linear polarized plane waves is used to represent the field inside the waveguide

$$\vec{E}_1 = Ee^{j\omega t - j\beta_x x - j\beta_z z}\vec{e}_y \tag{13}$$

$$\vec{E}_2 = -Ee^{j\omega t + j\beta_x x - j\beta_z z}\vec{e}_y. \tag{14}$$

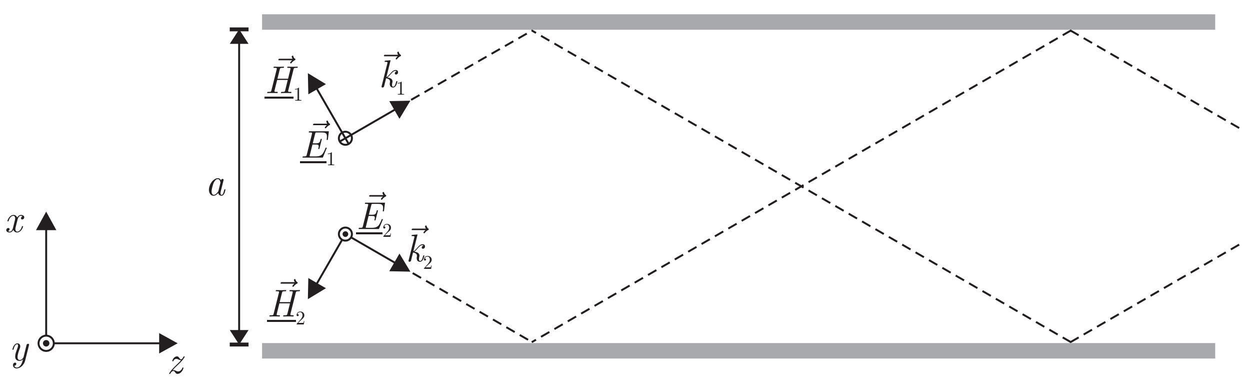

Figure 2: Parallel plate waveguide

a) Compute the complete field in the waveguide.

b) Determine β_x using the boundary conditions.

c) Determine β_z using the relation for plane waves $\beta_x^2 + \beta_z^2 = \omega^2 \mu \epsilon$.

Solution:

a) The superposition of the two waves results in

$$\vec{E}_{total} = \vec{E}_1 + \vec{E}_2 = -2jEe^{j\omega t - j\beta_z z} \sin(\beta_x x)\vec{e}_y. \tag{15}$$

b) The boundary conditions impose that

$$\vec{E}_{total}(x = 0, a) = 0 \rightarrow \beta_x = \frac{n\pi}{a}. \tag{16}$$

c) Using the given plane waves the wave number β_z yields to

$$\beta_z = \sqrt{\omega^2 \mu \epsilon - \left(\frac{n\pi}{a}\right)^2}. \tag{17}$$

In fig. 3 the relation given by (17) is depicted for different n.

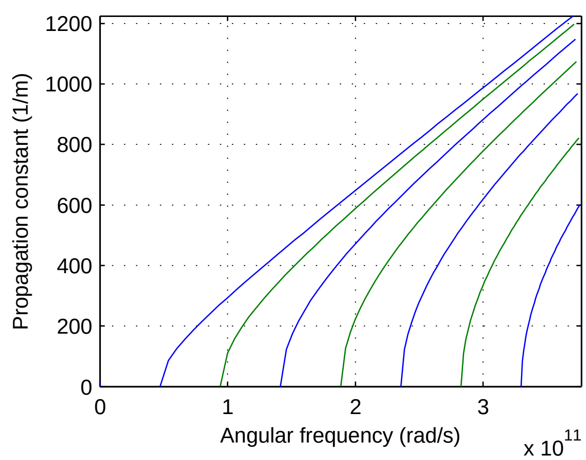

Figure 3: Relation between frequency and propagation constant β_z.