Exercise[1] 7

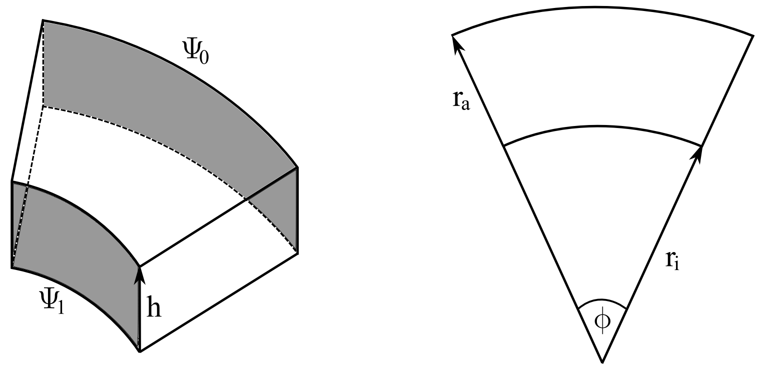

Figure 1: sketch of cylindrical sector resistor.

A three dimensional resistor of homogenous conductivity $\sigma = 1e7\ \frac{\text{S}}{\text{m}}$ is clamped between two electrodes with a potential difference of $\Psi_1 - \Psi_0 = 10\text{V}$ as depicted in figure 1 with the following parameter:

$$\phi = \frac{\pi}{4}, \qquad r_a = 5\text{ cm}, \qquad \text{r}_\text{i} = 2\text{ cm}, \qquad \text{h} = 1\text{ cm}.$$

For the following calculations neglect the fringing fields of the setup and assume a time invariant (static) electric fields.

a) Derive the governing partial differential equation from the Maxwell equation!
hint: $\nabla \cdot \left(\nabla \times \vec{A}\right) = 0, \qquad \nabla \times (\nabla\alpha) = 0$

b) What are the boundary conditions for each of the six areas?

c) Determine analytically the field potential $\Psi(\rho, \phi, z)$ within the resistor. Utilise the symmetric structure of the problem for simplified calculations.
hint: $\nabla^2\Psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{\partial^2\Psi}{\partial z^2}$

d) Compute the current through the resistor!

e) Compute overall resistance R of the resistor!

f) Solve the problem by Comsol. Compare the solution of Comsol with the analytical solution you obtained in part c! (This part was not asked in the exam)

Solution

a) Gauss' Law states that:

$$\nabla \cdot E = \frac{\rho_f}{\varepsilon_0} \tag{1}$$

where ρ_f is the free charge distribution in the space. For our question, there are no free charges therefore the right hand side of Eq.1 becomes 0. With the introduction of electrical potential,

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i.e. $E = -\nabla\Psi$, we get the following equation (Laplace's equation) that describes the potential distribution inside the conductor:

$$-\nabla \cdot (\nabla\Psi) = -\nabla^2\Psi = 0 \tag{2}$$

b) On the surfaces, where the potential difference is given, the boundary conditions (B.Cs) are:

$$\begin{aligned} \Psi &= \Psi_0 + 10(\textbf{on the surface at } \rho = 2cm.) \\ \Psi &= \Psi_0(\textbf{on the surface at } \rho = 5cm.) \end{aligned} \tag{3}$$

clearly, these are Drichlet B.Cs. In all the other faces, since there is no fringing field/current allowed, the normal component of the current density must be zero. Mathematically:

$$\mathbf{n} \cdot \mathbf{J} = 0 \tag{4}$$

where $\mathbf{J}$ is the current density and is related to the E-field by $\mathbf{J} = \sigma\mathbf{E}$. Again by introducing the electric potential into Eq.4, the following B.Cs are obtained for the other faces:

$$-\sigma\frac{\partial\Psi}{\partial\mathbf{n}} = 0 \tag{5}$$

which are Neumann BCs.

c) Inspecting the geometry reveals that the potential distribution is only the function of the radial distance, since the geometry is symmetric in all the other directions. Therefore, the partial derivatives with respect to φ and z will vanish and the Laplacian will be simplified to:

$$\nabla^2\Psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi}{\partial\rho}\right) \tag{6}$$

The solution of Eq.6 is given by:

$$\Psi(\rho) = a\ln(\rho) + b \tag{7}$$

where a and b are the constants to be determined by using the BCs. Combining the BCs given in Eq.3 and Eq.7 gives the following solution for the problem:

$$\Psi(\rho) = -10.9135\ln(\rho) - 32.6941 + \Psi_0 \tag{8}$$

Note that if the fringing fields were allowed, the simplifications in Eq.6 would not be possible.

d) To obtain the total current, we need to integrate the current density ($\mathbf{J} = \sigma\mathbf{E} = -\sigma\frac{\partial\Psi}{\partial\rho} = \frac{10.9135\sigma}{\rho}\hat{\rho}$) on a surface that closes the area of the conductor, which is given by:

$$\mathbf{I} = \hat{\rho}\int_0^{\frac{\pi}{4}} \frac{10.9135\sigma}{\rho}\rho d\varphi dz = 8.5686e5\hat{\rho} \tag{9}$$

e) By using the current value obtained in part d, the resistance of the conductor is given by:

$$R = \frac{\Psi}{|\mathbf{I}|} = 1.1671e - 5\Omega \tag{10}$$