Exercise 13:

A system of coupled differential equations is given by

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial f}{\partial t}\\ \frac{\partial f}{\partial x} &= u\end{aligned}$$

Answer the following questions:

a) Decouple the system of differential equations.

b) Find a FD update scheme for the decoupled differential equation for f using central differences for derivatives towards x and forward differences for derivatives towards t.

c) Find a FD update scheme for the coupled differential equations. Use central differences for derivatives towards x and forward differences for derivatives towards t.

d) Under which condition are schemes b) and c) equivalent?

e) Show that the accuracy of the used forward derivatives is $O(\Delta t)$.

f) Improve scheme b) such that it shows $O(\Delta t^2)$ and $O(\Delta x^2)$ accuracy.

Solution

a) The decoupled versions of the two equations are:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}\\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial f}{\partial t}\end{aligned} \tag{1}$$

b) The update scheme is:

$$\frac{f(x+\Delta x,t)-2f(x,t)+f(x-\Delta x,t)}{(\Delta x)^2} = \frac{f(x,t+\Delta t)-f(x,t)}{\Delta t} \tag{2}$$

c) The update scheme is:

$$\begin{aligned}\frac{u(x+\Delta x,t)-u(x-\Delta x,t)}{2\Delta x} &= \frac{f(x,t+\Delta t)-f(x,t)}{\Delta t}\\ \frac{f(x+\Delta x,t)-f(x-\Delta x,t)}{2\Delta x} &= u(x,t)\end{aligned} \tag{3}$$

d) When $\Delta t_b = \Delta t_c$ and $\Delta x_b = 2\Delta x_c$, both of the methods are the same.

e) By Taylor series expansion we have:

$$f(x, t + \Delta t) = f(x, t) + \frac{\Delta t}{1!}\frac{\partial f}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 f}{\partial t^2} + HOT \tag{4}$$

Now if we subtract $f(x, t)$ from the both sides of the expansion above and divide both sides by Δt, we obtain:

$$\frac{f(x + \Delta x, t) - f(x, t)}{\Delta x} = \frac{\partial f}{\partial t} + \frac{(\Delta t)}{2!}\frac{\partial^2 f}{\partial t^2} + HOT \tag{5}$$

From above we can see the error in the forward differences is $O(\Delta t)$

f) If we use central differences for both t and x we get the accuracy of $O(\Delta t^2)$ and $O(\Delta x^2)$

Exercise 14:

Given is Poisson's equation $\frac{d^2\Phi}{d^2x} + \frac{d^2\Phi}{d^2y} = f(x,y)$

a) What is the difference between Laplace's equation and Poisson's equation?

b) Give two examples where the Poisson equation is used to describe physics and indicate the meaning of the involved fields.

c) Write the Dirichlet and Neuman boundary conditions for the potential Φ.

d) What physical conditions are given by Dirichlet and Neuman boundaries in the two examples of b)? (Give for each physics a practical example of a Dirichlet and a Neuman boundary condition!)

e) Approximate the Laplace's equation with the FD method! The mesh distances are h_x and h_y.

f) Explain the staircasing effect in FD?

Solution

a) Laplace's equation is $\frac{d^2\Phi}{d^2x} + \frac{d^2\Phi}{d^2y} = 0$ and Poisson's equation is $\frac{d^2\Phi}{d^2x} + \frac{d^2\Phi}{d^2y} = f(x,y)$. This means Poisson's equation is more general. The solution of Laplace's equation is the homogenous part of the solution of Poisson's equation.

b) Static thermal simulation and electrostatics use Poisson's equation.

c) Dirichlet: $\Phi = g_1$. Neuman: $\frac{d\Phi}{dn} = g_2$.

d) Thermal: Dirichlet boundaries are boundaries with fixed temperature. Example: connection to a big thermal mass. Neuman boundaries are boundaries with constant heat flux. Example: surface of a body towards a medium with convection.
Electric: Dirichlet boundaries are boundaries with a given potential. Example: Metallic surface. Neuman boundaries are boundaries with constant E-field. Example: Symmetry boundary for an electrostatics problem with symmetric charge distribution.

e) $(\Phi(x+h_x,y) - 2\Phi(x,y) + \Phi(x-h_x,y))h_y^2 + (\Phi(x,y+h_y) - 2\Phi(x,y) + \Phi(x,y-h_y))h_x^2 = 0$

f) The staircasing effect occurs when domain boundaries are not parallel to the meshing directions x and y. The domain boundary has to take the form of staircases.