

## 1. Electrostatic Field

### 1.1. Introduction

A electrostatic field is a electric field produced by static electric charges. The charges are static in the sense of charge amount (it is constant in time) and their positions in space (charges are not moving relatively to each other). Due to its simple nature, the electrostatic field or its visible manifestation – electrostatic force - has been observed long time ago. Even ancient Greeks knew something about a strange property of amber that attracts (under certain conditions) small and light pieces of matter in its vicinity. Much later this phenomenon has been understood and explained as an effect of the electrostatic field. From this historical viewpoint, it would be logical to start the presentation of electromagnetic field theory with electrostatic field. Another reason, as it will be later clear, is its simplicity but also applicability. Namely, electrostatic field plays an important role in modern design of electromagnetic devices whenever a strong electric field appears. For example, an electric field is of paramount importance for the design of X-ray devices, lightning protection equipment and high-voltage components of electric power transmission systems [1], and hence an analysis of electrostatic field is needed. This is not only important for high-power applications. In the area of solid-state electronics, dealing with electrostatics is inevitable. It is sufficient to mention only the most prominent examples, such as resistors, capacitors or bipolar and field-effect transistors. Concerning computer and other electronic equipment, the situation seems to be similar: cathode ray tubes, liquid crystal display, touch pads etc. In order to reach a high level of knowledge and understanding needed for simulation of those practical 3D devices one has first to master the fundamental theory behind electrostatic field that will be presented in this chapter.

### 1.2. Coulomb's Law; Electric Field; Gauss's Law

An entire set of fundamental laws of electromagnetic field theory are the result of observation and experiment. The same empirical root brought about the basic law of electrostatic theory – Coulomb's<sup>1</sup> law [2, 3]. This law explains the electrostatic force between two point charges at certain distance in free space, as it is depicted in Figure 1.1.

French physicist Coulomb constructed a device for the precise measurement of the interaction forces between electric charges [2]. After extensive experiments he had noticed, mathematically formulated and finally published in the year 1785 the following law [3]:

$$\vec{F}_{12} = k \frac{Q_1 \cdot Q_2 \cdot (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \quad (1.1)$$

Basically, equation (1.1) says that the electrostatic force between two point charges is proportional to the amount of electrostatic charge on each of them and proportional to one over the square of their distance. The proportionality constant  $k$  depends on the unit system used and in the SI<sup>2</sup> interpretation (MKSA<sup>3</sup>) it takes the following form:

$$k = \frac{1}{4\pi\epsilon_0} \quad (1.2)$$

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<sup>1</sup> Charles Auguste de Coulomb (1763-1806), French physicist

<sup>2</sup> Le Système international d'unités (SI), Le Bureau international des poids et mesures (BIPM), Paris, France

<sup>3</sup> MKSA is an abbreviation of Meter-Kilogram-Second-Ampere

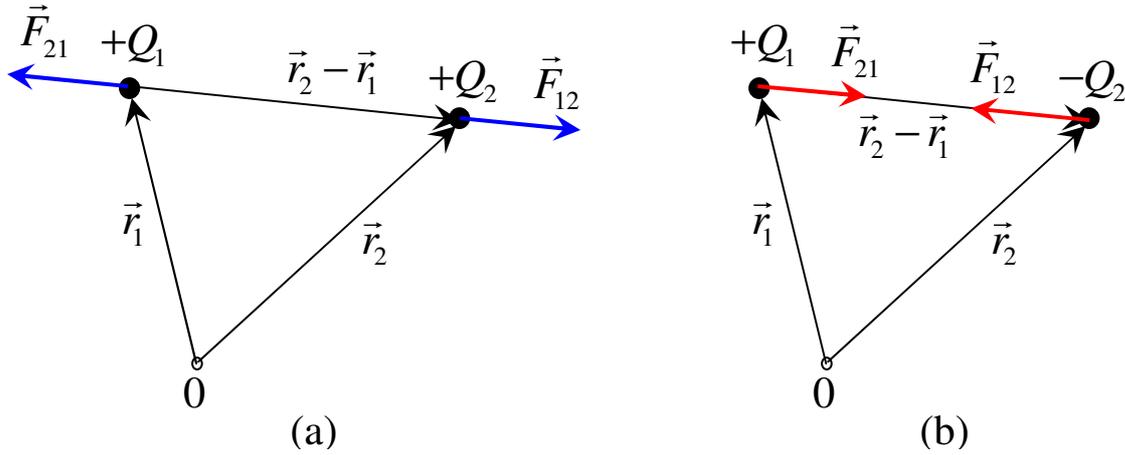


Figure 1.1. Coulomb's law;  $Q_1, Q_2$  are point charges;  $\vec{r}_1, \vec{r}_2$  are position vectors;  $\vec{F}_{12}$  is the electrostatic force acting on the charge  $Q_2$  due to the charge  $Q_1$  and  $\vec{F}_{21}$  is of other way around; Blue coloured force vectors represent the rejection (the same sign of charges) and red colour represents attraction ( the opposite sign of charges).

where  $\epsilon_0 = (8.85418788 \pm 0.00000007) \cdot 10^{-12}$  As/Vm is the dielectric permittivity of vacuum.

In order to emphasize the importance of Coulomb's discovery, in the SI system the unit for charge is chosen to be one Coulomb (C).

The force (1.1) appears only due to the existence of electric charges on both geometric objects. This means that around each electric charge the space has special tense condition that is observable through the force existence. This special feature or, so to speak, behaviour of the space around electric charges is called the *Electrostatic Field* or simpler the *Electric Field*. The force acting on the charge  $Q_1$  appears due to the electric field of charge  $Q_2$  at the position of charge  $Q_1$ . Mathematically written, it looks as follows:

$$\vec{E}_2 = \frac{\vec{F}_{21}}{Q_1} = \frac{Q_2 \cdot (\vec{r}_1 - \vec{r}_2)}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_2|^3} \quad (1.3)$$

Using equation (1.3) one can draw some more general and important conclusions. Namely, according to equation (1.3) (which has been always up to now experimentally confirmed) the electric field around any point electric charge has the form:

$$\vec{E}(\vec{r}) = \frac{Q \cdot (\vec{r} - \vec{r}_Q)}{4\pi\epsilon_0 |\vec{r} - \vec{r}_Q|^3} \quad (1.4)$$

where  $Q$  is the charge,  $\vec{r}_Q$  is the position vector of point charge and  $\epsilon_0$  is the dielectric constant or electric permittivity of a vacuum in which everything happens, i.e. the equations (1.3) and (1.4) are valid as such in free space (vacuum). So, the electric field in the space around point charge has spherical symmetry, depends on the charge amount, and decays as fast as  $1/r^2$ . The vectors of electric field are oriented towards the charge if  $Q < 0$  and for  $Q > 0$  the orientation is outwards.

If we have a system of  $N$  point charges distributed in free space (free space is a linear material in the dielectric sense because the dielectric permittivity of vacuum does not depend on the electric field) the *superposition principle* can be applied. It means that the total electric field can be calculated as the vector sum of all individual fields, as follows:

$$\vec{E}(\vec{r}) = \sum_{i=1}^N \frac{Q_i \cdot (\vec{r} - \vec{r}_i)}{4\pi\epsilon_0 |\vec{r} - \vec{r}_i|^3} \quad (1.5)$$

Furthermore, by following the same idea, a modified equation (1.5) can be used for a field computation of a continuous spatial charge distribution. This distribution is described usually by various charge density spatial functions. If the charge is distributed in a certain volume then we speak about the volume charge density (in  $C/m^3$ ), as follows:

$$\rho(\vec{r}) = \lim_{\Delta V \rightarrow 0} \frac{\Delta Q}{\Delta V}(\vec{r}) = \frac{dQ}{dV}(\vec{r}) \quad (1.6)$$

Electrostatic charge can be distributed also over a certain surface. Thus we speak about the surface charge density (in  $C/m^2$ ):

$$\sigma(\vec{r}) = \lim_{\Delta S \rightarrow 0} \frac{\Delta Q}{\Delta S}(\vec{r}) = \frac{dQ}{dS}(\vec{r}) \quad (1.7)$$

The remaining option would be the line charge density (in  $C/m$ ):

$$q(\vec{r}) = \lim_{\Delta L \rightarrow 0} \frac{\Delta Q}{\Delta L}(\vec{r}) = \frac{dQ}{dL}(\vec{r}) \quad (1.8)$$

Concerning for example the volume charge density, each small element of volume  $V$  that we usually denote as  $dV$  can be considered as a point charge carrying the amount of charge as big as  $dQ(\vec{r}') = \rho(\vec{r}') \cdot dV(\vec{r}')$ . Thus, by following the same logic as in (1.5), the electrostatic field of the spatial volume charge distribution can be calculated as:

$$\vec{E}(\vec{r}) = \iiint_{(V)} \frac{\rho(\vec{r}') \cdot (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3} dV(\vec{r}') \quad (1.9)$$

Similar to the previous equation for volume charge distribution, one can write the field of surface and line charge densities, respectively:

$$\vec{E}(\vec{r}) = \iint_{(S)} \frac{\sigma(\vec{r}') \cdot (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3} dS(\vec{r}') \quad (1.10)$$

$$\vec{E}(\vec{r}) = \iint_{(L)} \frac{q(\vec{r}') \cdot (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3} dL(\vec{r}') \quad (1.11)$$

A careful analysis of equation (1.4) shows that the electric field depends on the dielectric properties of the material due to the existence of the dielectric constant  $\epsilon_0$ . For practical reasons it would be useful to define a vector field that is independent of the medium:

$$\vec{D}(\vec{r}) = \epsilon_0 \vec{E}(\vec{r}) \quad (1.12)$$

The vector  $\vec{D}$  is called *the electric displacement* or *the electric flux density*. The importance of this vector can be seen from the formulation of Gauss's<sup>4</sup> Law, which constitutes along with the Coulomb's Law the mathematical basis of electrostatic field theory. Namely, the Gauss's Law says that the flux of the vector of electric displacement through any geometrically closed surface is equal to the total charge enclosed by that surface. The mathematical description of it has the following form:

$$Q_\Omega = \oint_{(\partial\Omega)} \vec{D}(\vec{r}) \cdot d\vec{S}(\vec{r}) = \iiint_{(\Omega)} \rho(\vec{r}) \cdot dV(\vec{r}) \quad (1.13)$$

Using the Divergence theorem [4], equation (1.13) leads us to the differential form of Gauss's law:

$$\nabla \cdot \vec{D}(\vec{r}) = \rho(\vec{r}) \quad (1.14)$$

which is one of the four famous Maxwell equations.

In the examples related to this chapter, it will be shown that the integral form (1.13) along with the differential form (1.14) are very important for the analytical computation of the electrostatic field in the case of highly symmetric problems.

### 1.3. Scalar Potential

For a given spatial distribution of the charge one can, using equation (1.9), compute the electric field at any point in the space. It is obvious that this procedure involves the integration of a vector function over the space where the electric charge exists. Due to convenient mathematical properties of the electrostatic field partially<sup>5</sup> described by equation (1.14) the static field computation can be simplified and reduced to the computation of some unknown scalar field instead of the unknown vector field. Namely, it is well known from vector calculus [5] that any vector field with curl operator identically equal to zero<sup>6</sup> and divergence operator not equal to zero at some points in the space can be represented as the gradient of some other scalar field. Such a scalar field is then called the *scalar potential* of the vector field, which in the case of the electrostatic field has the following form:

$$\vec{E}(\vec{r}) = -\nabla \Phi(\vec{r}) \quad (1.15)$$

where  $\Phi(\vec{r})$  is the scalar potential of the electric field. Using a well known identity from vector calculus [6]:

$$\nabla_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} = -\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad (1.16)$$

<sup>4</sup> Carl Friedrich Gauss (1777-1855), German mathematician, physicist and astronomer.

<sup>5</sup> Full description would be the equation (1.14) along with the equation  $\nabla \times \vec{E} = 0$  that will be derived later.

<sup>6</sup>  $\nabla \times \vec{a}(\vec{r}) \equiv 0$ , meaning that the curl operator of vector  $\vec{a}$  is identically equals to zero, i.e. it is everywhere in the space equals to zero.

and from equation (1.11) one can obtain the formula for scalar potential computation if the electric charge distribution is known, in the following form:

$$\Phi(\vec{r}) = \iiint_{(V)} \frac{\rho(\vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} dV(\vec{r}') \quad (1.17)$$

Using formulas (1.10) and (1.11) for a surface and a line charge distribution, in the same way it is possible to obtain the following:

$$\Phi(\vec{r}) = \iint_{(S)} \frac{\sigma(\vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} dS(\vec{r}') \quad (1.18)$$

$$\Phi(\vec{r}) = \int_{(L)} \frac{q(\vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|} dL(\vec{r}') \quad (1.19)$$

The electric potential can be obtained by integration of a simpler scalar function in (1.17), as opposed to the integration of a vector function in (1.11). Having the electric potential makes it very simple to compute the electric field using (1.15). It is obvious that the scalar potential makes things a lot simpler. At this point an interesting question would be: *What is the physical meaning of the scalar electric potential?* In order to find the answer to this question, let us consider the problem described in Figure 1.2.

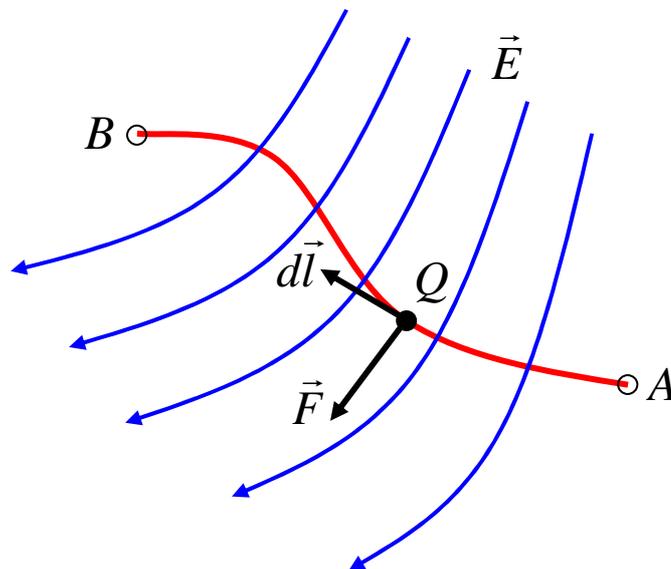


Figure 1.2. On the physical meaning of the scalar electric potential.  $Q$  is a point electric charge. Red line A-B is the path along which the charge  $Q$  moves in the electric field  $\vec{E}$ ; Blue lines are so called field lines<sup>7</sup> of the vector field  $\vec{E}$ ;  $\vec{F}$  is the electric force acting on the charge  $Q$ .

The problem is rather simple. The point charge  $Q$  is located somewhere in the space within an existing electric field. If one defines the curved path between two non coincident points  $A$  and  $B$  it is possible to calculate the work done in moving the charge against the force field in the

<sup>7</sup> The field line of a certain vector field is a curve in the space with the following property: the corresponding vector field is tangent the field line at every single point of the line.

following way. According to the equation (1.3), the electrostatic force at any point in the space can be calculated as:

$$\vec{F}(\vec{r}) = Q \cdot \vec{E}(\vec{r}) \quad (1.20)$$

The work that is done against this force by moving the charge along the path AB is:

$$W_{AB} = -\int_A^B \vec{F}(\vec{r}) \cdot d\vec{l}(\vec{r}) \stackrel{(1.18)}{=} -Q \cdot \int_A^B \vec{E}(\vec{r}) \cdot d\vec{l}(\vec{r}) \quad (1.21)$$

By taking into account (1.15) the work becomes:

$$W_{AB} = -Q \cdot \int_A^B \vec{E}(\vec{r}) \cdot d\vec{l}(\vec{r}) \stackrel{(1.15)}{=} Q \cdot \int_A^B \nabla \Phi(\vec{r}) \cdot d\vec{l}(\vec{r}) = Q \cdot \int_A^B d\Phi(\vec{r}) = Q \cdot (\Phi_B - \Phi_A) \quad (1.22)$$

Simple analysis of the equation (1.22) shows that  $Q \cdot \Phi$  can be interpreted as the potential energy of the charge  $Q$  at some position in electrostatic field, i.e. the scalar electric potential at certain point in space can be interpreted as the potential energy of the unit charge at this point in space with an existing electrostatic field. The second conclusion coming out of (1.22) is also very important. Namely, eventually it turns out that the shape and length of the path along which the charge has been moved plays no role in the final result<sup>8</sup>. If it happens that the points A and B have the same position then the total work equals to zero, no matter how long and complicated the curve really was. This can be expressed in a mathematical form as:

$$\oint_{(C)} \vec{E}(\vec{r}) \cdot d\vec{l}(\vec{r}) = 0 \quad (1.23)$$

which is essentially an equivalent form of the following equation:

$$\nabla \times \vec{E}(\vec{r}) = 0 \quad (1.24)$$

This can be easily proved by using the Stokes's theorem from vector calculus [7]. Any vector field that satisfies (1.23) is called a *potential* or *conservative field*.

#### 1.4. Electric Field in Media; Dielectric Polarization

Up to now we have considered electrostatic field in free space (vacuum). The existence of the electrostatic field is not related only to free space. It can exist also in various different materials. In an electrical sense materials are broadly classified as *conductors* and *non-conductors*. Non-conductors are sometimes called *insulators* or *dielectrics*. The basic property of a conductor is that the electric charge can freely move within it due to external electric field. Having the capability of free movement, the charges in a conductor will take such position that its internal field will cancel any applied external electrostatic field. Consequently, the electrostatic field inside conductor *is assumed to be zero*. As opposed to conductors, dielectrics are such materials in which movement of charge is not possible as long as the electric field intensity is below certain limit (the break-down of insulation). Although

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<sup>8</sup> Only the start (A) and end (B) points of the path are important.



polarisation at macroscopic level. Hence the vector of electric polarisation  $\vec{P}$  was introduced as a dipole moment volume density by the following equation:

$$\vec{P}(\vec{r}) = \lim_{\Delta V \rightarrow 0} \frac{\sum \vec{p}_i}{\Delta V} = \frac{d\vec{p}}{dV} \quad (1.25)$$

The SI unit for the electric polarisation  $\vec{P}$  is the Coulomb over square meter (C/m<sup>2</sup>).

Let the centre of dipole be at  $\vec{r}'$  and the observation point at  $\vec{r}$ . The equation for electric potential at the observation point produced by a single dipole can be easily derived using (1.4), (1.15) and (1.16) and it is:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{\vec{p} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad (1.26)$$

Using the equation (1.26) and (1.25) it is possible to calculate the potential at the observation point in the vicinity of a certain polarised dielectric body  $\Omega \subseteq R^3$ :

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \iiint_{(\Omega)} \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV(\vec{r}') \quad (1.27)$$

The vector function under the volume integral in (1.27) can be transformed using (1.16) into the following form:

$$\frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \stackrel{(1.16)}{=} \vec{P}(\vec{r}') \cdot \nabla_{\vec{r}'} \frac{1}{|\vec{r} - \vec{r}'|} \stackrel{[8]}{=} \nabla_{\vec{r}'} \left[ \vec{P}(\vec{r}') \cdot \frac{1}{|\vec{r} - \vec{r}'|} \right] - \nabla_{\vec{r}'} \cdot \vec{P}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} \quad (1.28)$$

Using (1.28) the integral (1.27) can be transformed into more useful form:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \iiint_{(\Omega)} \nabla_{\vec{r}'} \left[ \vec{P}(\vec{r}') \cdot \frac{1}{|\vec{r} - \vec{r}'|} \right] dV(\vec{r}') - \frac{1}{4\pi\epsilon_0} \cdot \iiint_{(\Omega)} \nabla_{\vec{r}'} \cdot \vec{P}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} dV(\vec{r}') \quad (1.29)$$

On the right-hand side of (1.29) we obtained two volume integrals. Obviously, the first of them can be reduced to a surface integral using the Divergence Theorem [4]:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \oint_{(\partial\Omega)} \vec{P}(\vec{r}') \cdot \vec{n}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} dS(\vec{r}') - \frac{1}{4\pi\epsilon_0} \cdot \iiint_{(\Omega)} \nabla_{\vec{r}'} \cdot \vec{P}(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} dV(\vec{r}') \quad (1.30)$$

Careful analysis of the equation (1.30) shows that the electric potential produced by polarized volume of material has two kinds of sources. The first is the polarized charge distributed over the surface of material (the surface integral in (1.30)) and the second is the polarized charge distributed over the volume of material (the volume integral (1.30)). By comparing these two different terms in (1.30) with the equations (1.18) and (1.17) it is possible to obtain the formulas for computation of polarized surface and volume charge density respectively, as follows:

$$\sigma_p(\vec{r}) = \vec{P}(\vec{r}) \cdot \vec{n}(\vec{r}), \vec{r} \in \partial\Omega \quad (1.31)$$

$$\rho_p(\vec{r}) = -\nabla \cdot \vec{P}(\vec{r}), \vec{r} \in \Omega \quad (1.32)$$

Equation (1.31) tells us that the surface polarisation charge exists always when the polarisation vector exists. It is distributed over the surface of the material and can be easily calculated as a scalar product of the polarisation vector and the normal unit vector to the surface at certain point. On the other hand the equation (1.32) tells us that the volume polarisation charge exists at certain point if the gradient of polarisation vector exists there. It is obvious from (1.25) that the gradient of polarisation vector equals to zero in *homogenous materials*, i.e. the material that has uniform distribution of dipoles over the volume. Those dipoles react on external electric field in the same way. Hence, the gradient of the polarisation vector at any point in such material is equal to zero. Thus homogenous material has no volume polarisation charge, which is a very important conclusion.

Our short and simple analysis of the equations (1.31) and (1.32) shows that in the case of homogenous dielectric the polarized charge is located over the surface ( $\partial\Omega$ ) of dielectric domain ( $\Omega$ ) and in the volume it is equal to zero. The physical explanation of this effect is rather simple: the atoms or molecules in the neighbourhood cancel each other and only at the outer surface polarized charge remains in macroscopic sense not equal to zero, i.e. we have surface density<sup>10</sup> of polarized charge (1.31).

For *isotropic media*<sup>11</sup> the electric polarisation vector is parallel to electric field with the certain constant of proportionality as follows:

$$\vec{P}(\vec{r}, t) = \epsilon_0 \chi_e \vec{E}(\vec{r}, t) \quad (1.33)$$

where are:

$\epsilon_0 = (8.85418788 \pm 0.00000007) \cdot 10^{-12}$  As/Vm – already mentioned the dielectric permittivity of vacuum

$\chi_e$  - electric susceptibility of the material

After all these relatively simple explanations related to the nature of polarisation we know that the vector  $\vec{P}$  contains the information about the response of a certain dielectric material to the external electric field. If the material is absent, i.e. if we consider the vacuum as our domain with an electric field then a logical conclusion would be that the vector  $\vec{P}$  is equal to zero. Furthermore, based on equation (1.12) and equation (1.14) one can obtain something very useful and of great practical importance. Namely, moving from vacuum towards dielectric materials, equation (1.12) should be modified such that the polarisation charge is taken into account. It can easily be done by using Gauss's law (1.14), but written for electric field:

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} + \frac{\rho_p(\vec{r})}{\epsilon_0} \quad (1.34)$$

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<sup>10</sup> See the equation (9)

<sup>11</sup> An isotropic media is a media that has the same properties in all directions

where  $\rho_p(\vec{r})$  represents the polarisation charge density. The next step of derivation would be to insert equation (1.32) into (1.34), which then becomes:

$$\nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} - \frac{1}{\epsilon_0} \nabla \cdot \vec{P}(\vec{r}) \quad (1.35)$$

or in a more appropriate form for our purposes:

$$\nabla \cdot [\epsilon_0 \vec{E}(\vec{r}) + \vec{P}(\vec{r})] = \rho(\vec{r}) \quad (1.36)$$

Equation (1.36), or more precisely its left-hand side, leads us to the conclusion that equation (1.12) has to be modified in order to fulfil the Gauss's law (1.14) in the case of polarized material in the following way:

$$\vec{D}(\vec{r}) = \epsilon_0 \vec{E}(\vec{r}) + \vec{P}(\vec{r}) \quad (1.37)$$

As it was already mentioned, the vector  $\vec{D}(\vec{r})$  is called *the electric displacement vector* or *electric induction* and its SI unit is the Coulomb per square meter (C/m<sup>2</sup>).

According to equations (1.37) and (1.33) one can draw an important conclusion:

$$\vec{D}(\vec{r}) = \epsilon_0 \vec{E}(\vec{r}) + \epsilon_0 \chi_e \vec{E}(\vec{r}) = \epsilon_0 (1 + \chi_e) \vec{E}(\vec{r}) = \epsilon \vec{E}(\vec{r}) \quad (1.38)$$

$$\epsilon = \epsilon_0 (1 + \chi_e) = \epsilon_0 \cdot \epsilon_r \quad (1.39)$$

where:

$\epsilon$  - *electric permittivity of material*,

$\epsilon_r$  - *relative permittivity or dielectric constant of material*.

## 1.5. Laplace's<sup>12</sup> and Poisson's<sup>13</sup> Equations

In the previous sections, the scalar potential has been introduced by (1.15) and the integral equations for its computation (1.17-1.19) have been given. As it will be shown later, besides the integral forms for field calculation, the differential forms play an even more important role in computation using either analytical or numerical approach. Consequently, it would be very useful at this stage to derive the differential equation of scalar potential, as well. In order to do so, one can recall again Maxwell's equation (1.14) written for electrostatics:

$$\nabla \cdot \vec{D}(\vec{r}) = \rho(\vec{r}) \quad (1.14)$$

Using equations (1.38) and (1.15), equation (1.14) becomes:

$$\nabla \cdot \vec{D}(\vec{r}) \stackrel{(1.38)}{=} \nabla \cdot (\epsilon(\vec{r}) \cdot \vec{E}(\vec{r})) \stackrel{(1.15)}{=} -\nabla \cdot (\epsilon(\vec{r}) \cdot \nabla \Phi(\vec{r})) = \rho(\vec{r}) \quad (1.40)$$

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<sup>12</sup> Pierre-Simon Laplace (1749-1827), French mathematician

<sup>13</sup> Siméon-Denis Poisson (1781-1840), French mathematician

Further modifications of (1.40) can be done by applying fundamental formulas of vector calculus [8, 9] as follows:

$$\nabla \varepsilon(\vec{r}) \cdot \nabla \Phi(\vec{r}) + \varepsilon(\vec{r}) \cdot \Delta \Phi(\vec{r}) = -\rho(\vec{r}) \quad (1.41)$$

The final form of (1.41) depends on the material properties, i.e. on the electric permittivity  $\varepsilon$ . With respect to the character of permittivity one can make the following rough classification of materials:

1. *Electrically anisotropic materials* are materials in which the vectors  $\vec{E}$ ,  $\vec{D}$  are not parallel. Consequently, the electric permittivity is not scalar but 3x3 *tensor* [11, 12], as follows:

$$\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \cdot \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix} \quad (1.42)$$

2. *Electrically inhomogeneous materials* are materials in which the electric permittivity is a function of position, i.e. electric properties of material are different in different points of space [11, 12]:

$$\varepsilon = \varepsilon(\vec{r}) \quad (1.43)$$

3. *Electrically nonlinear materials* are materials in which the electric permittivity depends on the electric field intensity [11, 12]:

$$\varepsilon = \varepsilon(\vec{E}) \quad (1.44)$$

Having in mind the classification of materials described by (1.42-1.44) one can easily see that for the simplest possible case of linear ( $\varepsilon(\vec{E}) = \text{const.}$ ), homogenous ( $\varepsilon(\vec{r}) = \text{const.}$ ) and isotropic ( $\vec{D} \parallel \vec{E}$ ) dielectric material, the equation (1.41) becomes way simpler:

$$\Delta \Phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon} \quad (1.45)$$

Equation (1.45) is called *Poisson's equation* [13]. If the right-hand side of (1.45) is equal to zero, i.e. if there is no free charge density in the domain of our interest, it becomes *Laplace's equation* [14]:

$$\Delta \Phi(\vec{r}) = 0 \quad (1.46)$$

Thus, having a solution for either equation (1.45) or (1.46), one can compute the electric field vector using equation (1.15). This is the basic advantage of electric potential. Instead of solving vector equations, it is sufficient to solve the scalar equation (1.46). The electric field computation afterwards is straightforward using the gradient of the potential function.

## 1.6. Electrostatic Boundary Value Problem

The problem of electrostatic field computation is considered to be solved if the potential and the field distribution are known in the domain of our interest. In order to compute these values one has to solve the Poisson's equation (1.45) if our domain is electrically homogenous, linear and isotropic. If it is not a case equation (1.45) will have different form. Obviously, the partial differential equation (1.45) alone can not uniquely describe the problem of electrostatic field, simply because it has infinite number of possible- so called partial solutions. In order to restrict the set of possible solutions to the solution of our concrete problem, additional conditions or constraints are needed. Those conditions are the field values over the boundary of the computational domain. Hence they are called the *Boundary Conditions*. In general there are various different types of boundary conditions in the electromagnetic field theory [11]. From electrostatic viewpoint it makes sense to speak about two fundamentally different types of boundary conditions:

1. Dirichlet<sup>14</sup> Boundary Condition, i.e. the boundary condition which defines the potential value over a certain part of boundary named as the Dirichlet Boundary  $\partial\Omega_D$  [15]:

$$\Phi(\vec{r}) = f_D(\vec{r}), \vec{r} \in \partial\Omega_D \quad (1.47)$$

2. Neumann<sup>15</sup> Boundary Condition, i.e. the boundary condition which defines the normal derivative of potential over a certain part of boundary denoted as the Neumann Boundary  $\partial\Omega_N$  [15]:

$$\frac{\partial\Phi}{\partial n}(\vec{r}) = f_N(\vec{r}), \vec{r} \in \partial\Omega_N \quad (1.48)$$

Having the Poisson's equation along with the Boundary conditions (1.47) and (1.48), it is possible now to write the Electrostatic Boundary Value Problem (BVP) in mathematically complete form:

$$\Delta\Phi(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon} \quad \vec{r} \in \Omega \subseteq R^3 \quad (1.49)$$

$$\Phi(\vec{r}) = f_D(\vec{r}) \quad \vec{r} \in \partial\Omega_D \quad (1.50)$$

$$\frac{\partial\Phi}{\partial n}(\vec{r}) = f_N(\vec{r}) \quad \vec{r} \in \partial\Omega_N \quad (1.51)$$

where  $\Omega$  is a computational domain and  $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$  is the boundary of our domain.

Although it is beyond the scope of this script, it is worth mentioning that there is a theorem for the existence and uniqueness of the solution of the electrostatic BVP described by (1.49-1.51) as in [10]. This means that the solution of every electrostatic problem with correctly defined boundary conditions exists and this solution is unique.

It is not necessary to have always both types of boundary conditions in the same time. Occasionally, it is possible to have only the Dirichlet type of boundary conditions (*the Dirichlet Problem*) or only the Neumann type of boundary conditions (*the Neumann Problem*)

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<sup>14</sup> Johann Peter Gustav Lejeune Dirichlet (1805-1859), German mathematician

<sup>15</sup> Franz Ernst Neumann (1798-1895), German mineralogist, physicist and mathematician

and of course both of them together but at different parts of boundary. It is important to mention that the solution of Neumann problem is not unique with respect to the scalar potential because only potential derivative has been prescribed (any constant can be added to the potential and solution is still correct). Since the electric field is equal to the gradient of electric potential, both problems (Neumann and Dirichlet) are unique with respect to the electric field.

The last remaining combination is the case when we have prescribed Dirichlet BC and Neumann BC over the same part of boundary:

3. *Cauchy*<sup>16</sup> *Boundary Condition*, i.e. a weighted average of the Dirichlet and Neumann BC [15]:

$$W_D \Phi(\vec{r}) + W_N \frac{\partial \Phi}{\partial n}(\vec{r}) = f_C(\vec{r}), \vec{r} \in \partial\Omega_C \quad (1.52)$$

where  $W_D$  and  $W_N$  are weighting factors (constant values).

Since both the Neumann and the Dirichlet problems are unique with respect to the electric field, the electrostatic problem involving Cauchy BC obviously can not be unique [10]. Hence the Cauchy problem in electrostatic has no practical importance.

After the analysis performed in the preceding part of this section, the electrostatic BVP is defined by (1.49-1.51). The question is: how to solve this problem? There are various different approaches to find its solution. In general two categories of solution methods with totally different approach and mathematical background can be clearly distinguished:

1. *Analytical Solution Methods*, based on the algorithms for analytical solution of partial differential equations, such as the Method of Images [10], the Method of Variables Separation [16], the Green's Function Method [10, 17-19] and so on. Those methods can be applied only to electrostatic problems with a high level of geometrical symmetry.
2. *Numerical Solution Methods*, based on the methods for the discretization of differential and/or integral operators in a computational domain (*Domain Methods*) and/or over its boundary (*Boundary Methods*), such as the Finite Element Method (FEM) [20, 21], the Methods of Moments (MOM) [22], the Boundary Element Method (BEM) [23-25], the Finite Difference Method (FDM) [26, 27] and so on. Numerical methods are general, i.e. can be applied for solution computing of the problems with an arbitrary complicated 3D geometry.

Although the analytical methods are very important for deep understanding of the electrostatic field phenomenon and sometimes even for the solution of some simple practical problems, they will not be described here. For further reading one can use some of already mentioned references. The basic target of this script is to give the students a detailed introduction into the numerical solution of various electromagnetic problems.

## 1.7. Graphical Representation of Electrostatic Fields

Let us for the moment consider the situation that the electrostatic field is solved in a certain domain of finite size. At this point, it does not matter which method for finding the solution has been used. It is important that the field values are known everywhere in the domain (or

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<sup>16</sup> Augustin Louis Cauchy (1789-1857), French mathematician

can be quickly computed from the discrete set of values that are available). Someone, who is interested in the solution of a certain electrostatic problem, can hardly find useful information when looking in a table of computed field values in various positions of the domain. It is obvious that some graphical representation is needed. Today in modern commercial software solutions for electromagnetic field computations, the electrostatic field is graphically presented in many various and fanciful ways. Usually, the average user of field simulation software has much more than he needs. Essentially, from a theoretical point of view one can clearly distinguish two major representations:

1. *Equipotential Surfaces* (3D) and *Equipotential Lines* (2D) are defined as a surfaces/lines over/along which the scalar potential has a constant value:

$$\Phi(x, y, z) = \text{const.} \quad (1.53)$$

2. *Field Lines* ( $\vec{E}$ -lines) and *Displacement Lines* ( $\vec{D}$  - lines) are lines along which, at any point, the vector  $\vec{E}$  and vector  $\vec{D}$  respectively are tangential vectors to the line. The mathematical expression for field line can be derived as follows:

$$d\vec{l} = dx \cdot \vec{e}_x + dy \cdot \vec{e}_y + dz \cdot \vec{e}_z, \quad \vec{E} = E_x \cdot \vec{e}_x + E_y \cdot \vec{e}_y + E_z \cdot \vec{e}_z \quad (1.54)$$

$$d\vec{l} \times \vec{E} = 0 \text{ - tangential vector condition} \quad (1.55)$$

Combining the equation (1.55) with (1.54), the governing equation of field lines is obtained:

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \quad (1.56)$$

It has been already shown by the equation (1.22) that the line integral of the electrostatic field over an arbitrary line is equal to the difference of potentials at starting and ending points of the curve. Hence this line integral is equal to zero along any equipotential line and due to the equation (1.22) the field lines and the equipotential lines are always **perpendicular** to each other.

In order to make this clearer let us consider for the moment the situation presented in Figure 1.4. Here we have a simple 2D electrostatic problem. Two metallic cylinders at opposite potential ( $\pm 1V$ ) with infinite length and parallel axes are placed close to each other in empty space (vacuum). Due to the cylinder's symmetry, one can easily compute the electric field around a single cylinder by using Gauss's law (1.13). Superposition of the fields of both cylinders gives the total electric field in the surrounding space. Having the field vectors and the equations of electrostatic fields that have been presented up to now one can easily produce the field lines and equipotential lines presented in Figure 1.4 using some graphical program. It is also possible to see that the equipotential lines (blue colour) are always perpendicular to the field lines (red colour). Since the electric field is a gradient of potential, one can, when looking at the potential line distribution, draw some conclusion about the electric field intensity. Namely, the regions with high electric field intensity are those regions where the potential lines are more densely distributed in the space, i.e. the potential value increases more quickly in space (larger gradient). Electric field is weaker in those regions where the equipotential lines are sparse (smaller gradient). Apparently, a closer look into Figure 1.4 shows that the electric field is strongest between the electrodes (metallic cylinders) and

decays with the distance from them.

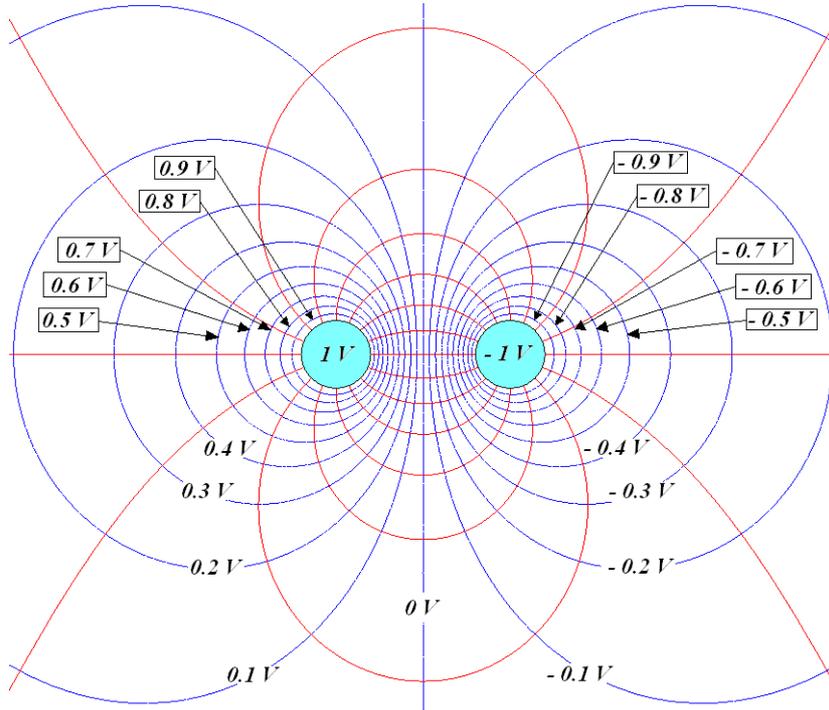


Figure 1.4. Graphical representation of the electrostatic field; Electrostatic field of two cylindrical electrodes at the potential of  $\pm 1V$  were analyzed; Red lines are the field lines; Blue lines are the equipotential lines;

### 1.8. Electrostatic Potential Energy; Electrostatic Force

Every spatial distribution of the electrostatic charge contains the amount of the potential energy that is equal to the work done by bringing them together in the final position against the electrostatic force acting between them. This energy balance is, of course an ideal case when losses of energy are not present. In order to derive the formula for electrostatic potential energy one has to be very careful because the electric field existing in the space is produced by the distributed charge itself. If the charge density at a certain point in the domain is described by the function  $\rho = \rho(\vec{r})$  it is useful to analyze the increase of the potential energy of the system if the charge density in a certain period of time increases its value with a very small amount  $d_t \rho(\vec{r})$ . This change of charge density is considered to be so small that the corresponding change of field and potential distribution is negligible. If the increase of charge density is considered as an additional charge that has been brought from infinity (the point of zero potential) to its current position then the increase of electrostatic potential energy according to the equation (1.22) can be calculated as:

$$d_t W_E = \int_{(\Omega)} d_t \rho(\vec{r}) \cdot \Phi(\vec{r}) \cdot dV(\vec{r}) \quad (1.57)$$

By introducing the equation (1.14) into (1.57), the following is obtained:

$$d_t W_E = \int_{(\Omega)} \nabla [d_t \vec{D}(\vec{r})] \cdot \Phi(\vec{r}) \cdot dV(\vec{r}) \quad (1.58)$$

Applying formula (12) from [28] for divergence of the product of scalar and vector function, the integral on the right-hand side can be written as:

$$d_t W_E = \int_{(\Omega)} \nabla \left[ d_t \vec{D}(\vec{r}) \cdot \Phi(\vec{r}) \right] \cdot dV(\vec{r}) - \int_{(\Omega)} d_t \vec{D}(\vec{r}) \cdot \nabla \Phi(\vec{r}) \cdot dV(\vec{r}) \quad (1.59)$$

The first volume integral can be transformed into a surface integral according to the Gauss's theorem [4]. In the second volume integral on the right-hand side the term  $-\nabla \Phi(\vec{r})$  appears which is the electric field. Thus the equation (1.59) becomes:

$$d_t W_E = \oint_{(\partial\Omega)} d_t \vec{D}(\vec{r}) \cdot \Phi(\vec{r}) \cdot \vec{n}(\vec{r}) \cdot dS(\vec{r}) + \int_{(\Omega)} \vec{E}(\vec{r}) \cdot d_t \vec{D}(\vec{r}) \cdot dV(\vec{r}) \quad (1.60)$$

where  $(\partial\Omega)$  is a boundary of the domain  $(\Omega)$ . The domain of integration is the entire space where the electrostatic field exists (total energy). Hence the first integral over boundary must be equal to zero because the electrostatic field of such a domain must vanish on the boundary. Furthermore, in order to get the total energy of the electrostatic field one has to integrate equation (1.60) in terms of time from an initial moment when the charge density everywhere was equal to zero, up until this very moment "t". Hence the electrostatic potential energy is in general:

$$W_E = \int_0^t d_t W_E = \int_{(\Omega)} \left\{ \int_0^t \vec{E}(\vec{r}) \cdot d_t \vec{D}(\vec{r}) \right\} \cdot dV(\vec{r}) \quad (1.61)$$

or in a more appropriate form:

$$W_E = \int_{(\Omega)} \left\{ \int_0^{D(t)} \vec{E}(\vec{r}) \cdot d_t \vec{D}(\vec{r}) \right\} \cdot dV(\vec{r}) \quad (1.62)$$

If the domain is linear, i.e. if the permittivity does not depend on the electric field ( $\varepsilon \neq \varepsilon(E)$ ), the equation (1.62) becomes much simpler:

$$W_E = \frac{1}{2} \int_{(\Omega)} \vec{E}(\vec{r}) \cdot \vec{D}(\vec{r}) \cdot dV(\vec{r}) \quad (1.63)$$

As one can see from the equation (1.63) the volume density of electrostatic field energy in dielectrically linear material is:

$$w_E = \frac{dW_E}{dV} = \frac{1}{2} \vec{E}(\vec{r}) \cdot \vec{D}(\vec{r}) \quad (1.64)$$

By following a similar procedure and using the basic electrostatic identities it is possible to derive a general formula for the electrostatic force computation. Let us consider again the dielectric domain  $\Omega \subseteq R^3$  that is homogenous with permittivity  $\varepsilon$ . The total force acting on that domain according to the equation (1.20) can be calculated as:

$$\vec{F}_{tot} = \int_{(\Omega)} \rho(\vec{r}) \vec{E}(\vec{r}) dV(\vec{r}) \stackrel{(1.14)}{=} \epsilon \int_{(\Omega)} \vec{E}(\vec{r}) \cdot \nabla \vec{E}(\vec{r}) dV(\vec{r}) \quad (1.65)$$

Equation (1.65) will lead us towards a very interesting conclusion. Namely, at this point we have to refresh our knowledge about one interesting vector identity [8]:

$$\int_{(\Omega)} \left[ \vec{a} \cdot \nabla \vec{b} + \vec{b} \cdot \nabla \vec{a} - \vec{a} \times (\nabla \times \vec{b}) - \vec{b} \times (\nabla \times \vec{a}) \right] dV = \oint_{(\partial\Omega)} \left[ \vec{a} (\vec{b} \cdot \vec{n}) + \vec{b} (\vec{a} \cdot \vec{n}) - (\vec{a} \cdot \vec{b}) \vec{n} \right] dS \quad (1.66)$$

It is not that difficult to see that by introducing (1.66) into (1.65) the electrostatic force becomes:

$$\vec{F}_{tot} = \epsilon \oint_{(\partial\Omega)} \left[ \vec{E} \cdot (\vec{n} \cdot \vec{E}) - \frac{1}{2} \vec{n} \cdot E^2 \right] dS = \oint_{(\partial\Omega)} \vec{T} \cdot dS \quad (1.67)$$

where  $\vec{T}$  represents the surface force density. The conclusion (1.67) is very interesting because it shows that the surface forces depend not only on the field intensity but also on the field orientation with respect to the surface element, i.e. to the normal vector. This is somehow an analogue situation to the stress of material in linear elasticity. As we will see later this idea led Maxwell to, the stress tensor of the electromagnetic field<sup>17</sup>. Equation (1.67) allows us to compute the total force acting on a certain volume as a surface integral over the boundary of that volume, which is sometimes very useful in practical field simulation problems.

## 1.9. Interface Conditions

Using Gauss's law (1.13) and the conservative characteristic of electrostatic fields (1.23) expressed in an integral form, one can draw important conclusions about the electrostatic field behaviour at the interface between different materials. Let us consider the situation depicted in Figure 1.5.

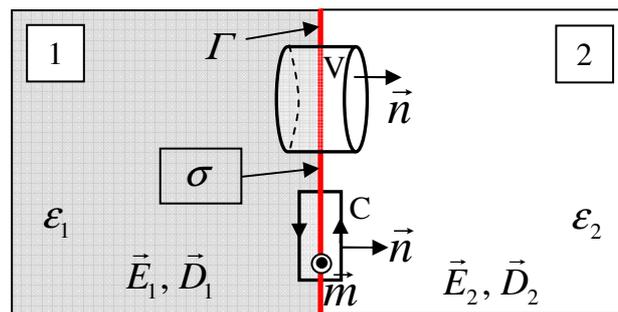


Figure 1.5. Interface between different materials (materials are different in dielectric sense);  $V$  – cylindrical small volume;  $C$  – small contour;  $\sigma$  – surface density of electric charge over interface;  $\Gamma$  – surface boundary (interface) between two domains.

Our analysis will start with the surface integral of vector  $\vec{D}(\vec{r})$  over the boundary  $\partial V$  of domain  $V$ . According to equation (1.13) from chapter 1.2, it is possible to write the following:

<sup>17</sup> Maxwell's stress tensor.

$$\oiint_{(\partial V)} \vec{D}(\vec{r}) \cdot \vec{n}(\vec{r}) dS(\vec{r}) = \iiint_{(V)} \rho(\vec{r}) dV(\vec{r}) \quad (1.68)$$

The cylindrical domain  $V$  is determined by the circular base  $\Delta S$  and the height. The base of cylinder  $\Delta S$  is small enough such that the field over that surface can be considered as constant, i.e. the left-hand side of (1.68) becomes:

$$\oiint_{(\partial V)} \vec{D}(\vec{r}, t) \cdot \vec{n}(\vec{r}) dS(\vec{r}) = \iint_{(\Delta S_1)} \vec{D}(\vec{r}) \cdot \vec{n}(\vec{r}) dS(\vec{r}) + \iint_{(\Delta S_2)} \vec{D}(\vec{r}) \cdot \vec{n}(\vec{r}) dS(\vec{r}) + \iint_{(S_{LAT})} \vec{D}(\vec{r}) \cdot \vec{n}(\vec{r}) dS(\vec{r}) \quad (1.69)$$

where  $S_1, S_2$  are bases of the cylinder belonging to the domain1 and domain 2 respectively and  $S_{LAT}$  is the lateral surface of the cylinder.  $h$ . In order to draw conclusions about the local field behaviour we will analyze the limiting process when the height  $h$  tends to zero. Thus, the area of lateral surface is  $\lim_{h \rightarrow 0} S_{LAT} = 0$ . Hence the third term on the right-hand side of (1.69) vanishes when the field under integral has finite value. The remaining terms on the right-hand side of (1.69) become much simpler due to the fact that the field over the bases  $\Delta S_{1,2}$  can be considered as constant:

$$\lim_{h \rightarrow 0} \oiint_{(\partial V)} \vec{D}(\vec{r}) \cdot \vec{n}(\vec{r}) dS(\vec{r}) = (\vec{D}_1 - \vec{D}_2) \cdot \vec{n} \Delta S \quad (1.70)$$

On the other hand, the right-hand side of (1.68) tends to be zero when  $h \rightarrow 0$  if the function  $\rho(\vec{r})$  has finite value inside of  $V$ . The surface charge density that has been already introduced by equation (1.7) can be considered as the singularity of the volume charge density. Hence one has to be very careful with the limit  $h \rightarrow 0$ :

$$\lim_{h \rightarrow 0} \iiint_{(V)} \rho(\vec{r}, t) dV = \left. \begin{array}{l} dV = h \cdot \Delta S \\ \lim_{h \rightarrow 0} [\rho(\vec{r}, t) \cdot h] = \sigma(\vec{r}, t) \end{array} \right| = \sigma \cdot \Delta S \quad (1.71)$$

By introducing the equations (1.70) and (1.71) into  $\lim_{h \rightarrow 0}$ (1.68) a conclusion of great importance is obtained:

$$\boxed{(\vec{D}_1 - \vec{D}_2) \cdot \vec{n} = \sigma} \quad (1.72)$$

In order to employ the equation (1.23) the line integration has to be performed over a closed loop  $C$  depicted in Figure 1.5. The size of the loop in the direction parallel to the boundary  $\Gamma$  is  $\Delta l$  which is small enough to consider the field over this part of the loop constant. The loop size perpendicular to the boundary is denoted as  $h$ . Similar to the previous analysis the limit  $h \rightarrow 0$  will be analyzed.

$$\oint_{(C)} \vec{E}(\vec{r}) \times \vec{n}(\vec{r}) dl(\vec{r}) = 0 \quad (1.73)$$

where  $S_C$  is the surface of contour  $C$ ,  $\vec{m}$  is a normal unit vector to the surface  $S_C$ . The left-hand side of (1.73) becomes:

$$\oint_{(C)} \vec{E}(\vec{r}) \times \vec{n}(\vec{r}) d\vec{r} = \int_{(\Delta_1)} \vec{E}(\vec{r}) \times \vec{n}(\vec{r}) d\vec{r} + \int_{(\Delta_2)} \vec{E}(\vec{r}) \times \vec{n}(\vec{r}) d\vec{r} + \int_{(h_{12})} \vec{E}(\vec{r}) \times \vec{n}(\vec{r}) d\vec{r} + \int_{(h_{21})} \vec{E}(\vec{r}) \times \vec{n}(\vec{r}) d\vec{r} \quad (1.74)$$

where:  $\Delta_1$  and  $\Delta_2$  are the parts of the loop  $C$  parallel to the boundary belonging to the domain 1 and domain 2, respectively; while  $h_{12}$  and  $h_{21}$  are the parts perpendicular to the boundary ( $h_{12}$  denotes the path from the domain 1 towards the domain 2 and for  $h_{21}$  is the other way around). Taking into account the limiting process  $h \rightarrow 0$  the third and fourth term on the right-hand side of (1.74) vanish due to the fact that integration path tends to be zero and field under integral must be finite. Hence the limit  $h \rightarrow 0$  of the equation (1.74) becomes:

$$\lim_{h \rightarrow 0} \oint_{(C)} \vec{E}(\vec{r}) \times \vec{n}(\vec{r}) d\vec{r} = \vec{E}_1 \times \vec{n}(\vec{r}) \Delta - \vec{E}_2 \times \vec{n}(\vec{r}) \Delta = 0 \quad (1.75)$$

$$\boxed{(\vec{E}_2 - \vec{E}_1) \times \vec{n}(\vec{r}) = 0} \quad (1.76)$$

Equations (1.72) and (1.76) are important interface conditions for the electrostatic field that have to be fulfilled over the boundary between two materials with different dielectric properties. As it will be shown later, they play a very important role in both analytical and numerical field computations.

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