2. Stationary Current Distribution

2.1. Introduction

In this chapter the electric field in conductors will be analyzed. Namely, in electrostatic case the electric charge was assumed to be fixed at certain position without possibility to move or change its amount. Here we will analyze what happens with the electric field inside a conductive domain where charge can move due to existing electric field, i.e. Coulomb's force. In stationary current we limit ourselves to the uniform movement of charge, i.e. all partial derivatives with respect to time are equal to zero.

2.2. Electric Current Density

Although in this chapter we present the stationary movement of electric charge we will give here a general definition of electric current and its density valid for any time-varying movement of charge. The volume charge density at certain position in space is a temporary value that will in general change in time, i.e. $\rho(\vec{r},t)$. In order to represent the movement of electric charge in 3D space and to predict its distribution in the future, we will introduce at this stage for us a new vectorial physical value called the density of electric current. We will do this with the help of Figure 1.

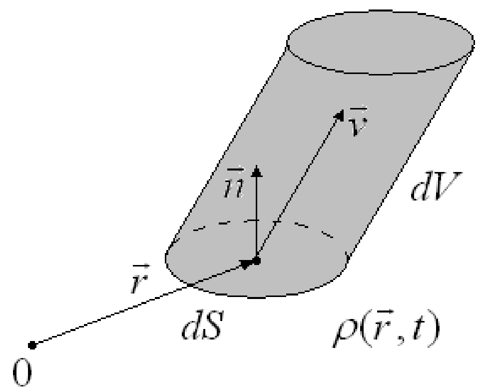

Figure 2.1. The definition of the vector of current density is presented.

Let us consider a single point in the space, determined by position vector $\vec{r}=(x,y,z)$, with associated electric charge density $\rho(\vec{r},t)$ and the velocity vector of charge movement $\vec{v}$. At this point we define the infinitely small circular surface $d\vec{S}=\vec{n}\cdot dS$ ($\vec{n}$ - normal unit vector of the surface) with the centre at the point $\vec{r}$. For an infinitely small time interval dt all points of this surface will be moved for the vector $d\vec{h}=\vec{v}\cdot dt$ creating thus the cylindrical volume dV (the grey volume in Figure 2.1). Accordingly the volume dV can be calculated in the following way [1, 2]:

$$dV = d\vec{S}\cdot d\vec{h} = \vec{n}\cdot\vec{v}\ dt\ dS \tag{2.1}$$

The total charge that went through the surface $d\vec{S}$ during the small time interval dt can be calculated as:

$$dQ = \rho\ dV = \rho\ \vec{n}\cdot\vec{v}\ dt\ dS \tag{2.2}$$

By dividing equation (2.2) with dt, the left hand side becomes very interesting. Namely, the well known term $I = d_t Q / dt$ [1] appears there [1, 2]. After the division the equation (2.2) becomes:

$$dI = \rho\, \vec{n} \cdot \vec{v}\; dS \tag{2.3}$$

On the left-hand side of equation (2.3) we have the infinitely small current dI because the current flows through the infinitely small surface dS. Further division of the equation (2.3) with dS yields the term dI/dS which is apparently the normal component of the vector $\rho\vec{v}$ to the surface dS. This vector is called the *volume density of current*, in our notation $\vec{J}$:

$$J_n = \rho\, \vec{n} \cdot \vec{v}\; dS = \vec{J} \cdot \vec{n} \tag{2.4}$$

Hence directly from (2.4) the vector of electric current density can be written as [1, 2]:

$$\vec{J}(\vec{r},t) = \rho(\vec{r},t)\, \vec{v}(\vec{r},t) \tag{2.5}$$

The SI unit of the current density is the Ampère per square meter (*A/m*^{2}). It is important here to clarify the fact that we have defined the volume current density and the unit is *A/m*^{2}. The name "*volume* current density" comes from the fact that this current flows in the volume of a certain body or domain and the unit *A/m*^{2} appears because the current density is related to the flow of charge through a certain surface. The name "*current* density" comes from the electric current that is a macroscopic scalar physical value defined as the electric charge flow per unit of time. The electric current is always related to some surface (usually this is a cross section of conductor) and according to (2.3), can be calculated using the current density as follows:

$$I(t) = \iint_{(S)} \vec{J}(\vec{r},t) \cdot \vec{n}(\vec{r})\; dS(\vec{r}) \tag{2.6}$$

In addition to the presented definition of *volume current density* represented by (2.5) it is possible to define the *surface current density*:

$$\vec{K}(\vec{r},t) = \rho_S(\vec{r},t)\, \vec{v}(\vec{r},t) \tag{2.7}$$

where $\rho_S(\vec{r},t)$ is a surface electric charge density. Although the surface charge and surface current are not so realistic on the first glance, they are rather close to some practical situations. As it will be shown later, the surface charge appears on the surface of a conductor in electrostatics and the surface current flows over the surface of conductor excited by source of electromagnetic fields working at high frequencies.

Although it is not directly visible from (2.5) the current density in a conductor is caused by the electric field inside of conductor. At the beginning of theoretical development of electromagnetics it was observed that the dependence between the current density and electric field is linear. This is called Ohm's[2] Law and this can be written as [1-3]:

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[1] The term $d_t Q / dt$ represents the charge flow per time unit which is a well known definition of the electric current I (SI system unit is Ampère – A)

[2] Georg Simon Ohm (1789-1854), German physicist

$$\vec{J}(\vec{r}) = \sigma \vec{E}(\vec{r}) \tag{2.8}$$

where σ is called the specific electric conductivity, SI unit ($A/(Vm)$). It represents the electrical properties of material.

2.3. Continuity Equation

A very important statement about the electric charge is the charge conservation law. This physical law states that charge can be neither created nor vanish. Practically, this means that the change of the amount of total charge Q in the domain $\Omega \subseteq R^3$ (3D space) can appear only due to a charge flow through the surface boundary $\partial\Omega$ (see Figure 2.2).

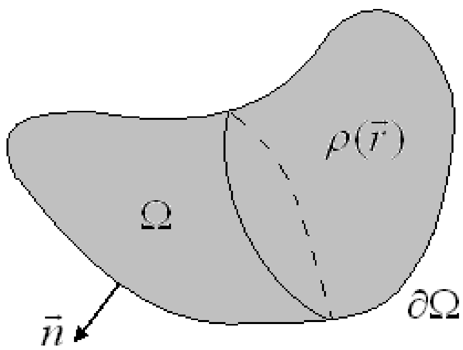

Figure 2.2. The 3D domain of finite size containing the charge smoothly distributed in the volume of domain according to the charge density function $\rho(\vec{r})$

According to the definition of electric current, the flow of electric charge towards outside of the domain (the electric current in direction of vector $\vec{n}$ normal to the surface $\partial\Omega$) produces the negative differential of total charge in the domain (less charge remains inside). It can be written in the following form:

$$I_{OUT}(t) = -\frac{\partial Q(t)}{\partial t} \tag{2.9}$$

By using the knowledge that we have gained in proceeding analysis about the current and charge density, it is possible to write:

$$\oint\!\!\!\oint_{(\partial\Omega)} \vec{J}(\vec{r},t) \cdot \vec{n}(\vec{r})\, dS(\vec{r}) = I_{OUT}(t) = -\frac{\partial Q(t)}{\partial t} = -\iiint_{(\Omega)} \frac{\partial \rho(\vec{r},t)}{\partial t}\, dV(\vec{r}) \tag{2.10}$$

Apparently, we have derived the charge continuity equation in the integral form:

$$\oint\!\!\!\oint_{(\partial\Omega)} \vec{J}(\vec{r},t) \cdot \vec{n}(\vec{r})\, dS(\vec{r}) = -\iiint_{(\Omega)} \frac{\partial \rho(\vec{r},t)}{\partial t}\, dV(\vec{r}) \tag{2.11}$$

If we want to obtain the charge continuity equation in differential form we have to apply Gauss's formula [4] or the Divergence theorem [5, 6] from vector calculus to the left hand side of the equation (2.11). Thus, we have the following:

$$\oint\!\!\!\oint_{(\partial\Omega)} \vec{J}(\vec{r},t)\cdot\vec{n}(\vec{r})\,dS(\vec{r}) = \iiint_{(\Omega)} \nabla\cdot\vec{J}(\vec{r},t)\,dV(\vec{r}) \tag{2.12}$$

In order for the right hand sides of equations (2.11) and (2.12) to be equal, their integrands must be equal. Therefore it is possible to write the following:

$$\boxed{\nabla\cdot\vec{J}(\vec{r},t) = -\frac{\partial\rho(\vec{r},t)}{\partial t}} \tag{2.13}$$

The equation (2.13) is called the charge continuity equation in differential form and has played very important role in a theoretical development of electromagnetic theory. As we have mentioned at the beginning of this chapter, for the stationary analysis, all time-derivatives are equal to zero. Hence the stationary continuity equation has the following form:

$$\boxed{\nabla\cdot\vec{J}(\vec{r}) = 0} \tag{2.14}$$

The equation (2.14) shows that the divergence of the vector of current density is equal to zero in the case of stationary fields, i.e. it simply says that the current lines (the line along which the current density vector is always tangent) must be closed lines[3] . The equation (2.14) helps us to derive one important additional conclusion. Namely if we employ Ohm's law (2.8) along with Maxwell's equation (1.14) we get the following:

$$\nabla\cdot\vec{J}(\vec{r}) \overset{(2.8)}{=} \nabla\cdot\left(\sigma\vec{E}(\vec{r})\right) = \sigma\,\nabla\vec{E}(\vec{r}) \overset{(1.37)}{=} \frac{\sigma}{\varepsilon}\nabla\vec{D}(\vec{r}) \overset{(1.14)}{=} \frac{\sigma}{\varepsilon}\rho \overset{(2.14)}{=} 0 \Rightarrow \boxed{\rho = 0} \tag{2.15}$$

So, the density of free charge in conductor is equal to zero for time-stationary fields. Therefore the field in a conductor must be created from outside.

2.4. Boundary Value Problem of Stationary Current Distribution

In order to analyze the stationary current distribution problem in more details let us consider the curl operator of the equation (2.8):

$$\nabla\times\vec{J}(\vec{r}) = \nabla\times\left(\sigma\vec{E}(\vec{r})\right) \tag{2.16}$$

For isotropic and homogenous conductor ($\sigma \neq \sigma(\vec{r})$) the right hand side becomes simpler. By using Maxwell's equation (1.24) it becomes as follows:

$$\sigma\,\nabla\times\vec{E}(\vec{r}) \overset{(1.24)}{=} 0 \tag{2.17}$$

Thus the curl operator of stationary electric field in conductor is equal to zero. This allows us to define here a scalar electric potential:

$$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r}) \tag{2.18}$$

[3] The current density lines have neither a source nor sink.

similar to the electrostatic field. According to equations (2.18), (2.17), (2.14) and (2.15) we can establish an analogy with electrostatics, presented in Table 2.1. Apparently we can use the same BVP for stationary current distribution as for electrostatics (1.49-1.51) with $\rho = 0$. This can be described with the following equations:

Table 2.1 The analogy between the Electrostatic and Stationary Current Distribution is presented.

Value	Electrostatic field	Stationary Current Distribution
Potential	Φ	Φ
Electric Field	$\vec{E} = -\nabla\Phi$	$\vec{E} = -\nabla\Phi$
Curl of Electric Field	$\nabla\times\vec{E} = 0$	$\nabla\times\vec{E} = 0$
Material Property	ε	σ
Associated Vector	$\vec{D} = \varepsilon\vec{E}$	$\vec{J} = \sigma\vec{E}$
Divergence of Associated Vector	$\nabla\cdot\vec{D}(\vec{r}) = \rho$	$\nabla\cdot\vec{J}(\vec{r}) = 0$

$$\Delta\Phi(\vec{r}) = 0 \qquad \vec{r}\in\Omega\subseteq R^3 \tag{2.19}$$

$$\Phi(\vec{r}) = f_D(\vec{r}) \qquad \vec{r}\in\partial\Omega_D \tag{2.20}$$

$$\frac{\partial\Phi}{\partial n}(\vec{r}) = f_N(\vec{r}) \qquad \vec{r}\in\partial\Omega_N \tag{2.21}$$

where Ω is a computational domain and $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ is the boundary of our domain. It is also easy to understand that the BC (2.20) describes the potential over the contacts of conductor (incoming and outgoing current must be equipotential surfaces). On the other hand the BC (2.21) represents the normal component of the current over certain surfaces (for example over the lateral surface of conductor it is $f_N(\vec{r}) = 0$).

Regarding interface conditions it is possible, using (1.72) and (1.76) from electrostatics and using Table 2.1, to write the following:

$$\left(\vec{J}_1 - \vec{J}_2\right)\cdot\vec{n} = \rho_S \tag{2.22}$$

$$\left(\frac{\vec{J}_2}{\sigma_2} - \frac{\vec{J}_1}{\sigma_1}\right)\times\vec{n}(\vec{r}) = 0 \tag{2.23}$$

The stationary current distribution analysis is used mainly to determine the current density in the conductor of complicated shape (for example the transmission line of printed board). This can be afterwards used to determine the resistance, heating, etc.

2.5. References

[1] M. Sadiku, "Elements of Electromagnetics", Saunders College Publishing, Fort Worth, 1989.

[2] J. D. Jackson, Classical Electrodynamics, John Wiley & Sons, New York, 1975.

[3] Ch. Hafner, *Post-modern Electromagnetics*, Wiley & Sons, New York, 1999.

[4] G. Arfken, "Gauss's Theorem", §1.11 in Mathematical Methods for Physicists, 3rd ed. Orlando, FL: Academic Press, pp. 57-61, 1985.

[5] Wolfram Math World, the web's most extensive mathematics resource, http://mathworld.wolfram.com/.

[6] Weisstein, Eric W. "Divergence Theorem." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/DivergenceTheorem.html