3. Magnetostatic Field

3.1. Introduction

A magnetostatic field is a magnetic field produced by electric charge in permanent uniform movement, i.e. in a permanent movement with constant velocity. Any directed movement of electric charge is called *Electric Current*. Hence the electric currents from a magnetostatic point of view are always constant in time. The theory of magnetostatic field has large practical significance. Its impact on everyday life is important. It is enough to mention some devices whose design is based on the analysis of magnetostatic fields: motors, transformers, TV cathode-tube deflection systems, magnetically levitated high-speed vehicles etc. Similar to electrostatic fields, there are two empirically obtained fundamental laws of magnetostatics (the Biot[16]-Savart[17] Law and Ampère's Law). A complete magnetostatic theory can be derived starting from those two fundamental laws.

3.2. Biot-Savart Law; Ampere's Law

In the early 1800s, French scientists Biot and Savart were trying to understand the relationship between electric current and magnetism. They had carried out many experiments and finally they discovered (roughly in 1820) that each element of a conductor[18] with electric current produces a magnetic field in its surrounding space according to the following equation:

$$d\vec{H}(\vec{r}) = \frac{I\, d\vec{l} \times \vec{r}}{4\pi r^3} \tag{3.1}$$

where I is current intensity, $d\vec{l}$ is the element of length of the conductor and $\vec{r}$ is the position vector of the point where the magnetic field is to be calculated taking the current element as the origin.

Equation (3.1) is clear, simple and general. Using equation (3.1) one can compute the magnetic field of a conductor with an arbitrary shape by integrating along the conductor. Hence, the magnetic field of an arbitrary conductor with electric current, according to (3.1) can be calculated as:

$$\vec{H}(\vec{r}) = \int_{(L)} \frac{I\, d\vec{l}\left(\vec{r}\,'\right)\times\left(\vec{r}-\vec{r}\,'\right)}{4\pi\left|\vec{r}-\vec{r}\,'\right|^3} \quad \text{(Line Current)} \tag{3.2}$$

The SI unit for magnetic field is the Ampère over meter (A/m).

Equation (3.2) describes the magnetic field of line current (wire). It can happen that current flows over the surface of a certain body. In this particular situation we speak about surface current density $\vec{K}(\vec{r})$ that produces, similar to (3.1), the following magnetic field:

$$\vec{H}(\vec{r}) = \int_{(S)} \frac{\vec{K}\left(\vec{r}\,'\right)\times\left(\vec{r}-\vec{r}\,'\right)}{4\pi\left|\vec{r}-\vec{r}\,'\right|^3} dS\left(\vec{r}\,'\right) \quad \text{(Surface Current)} \tag{3.3}$$

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[16] Jean-Baptiste Biot (1774 – 1862), French physicist and mathematician

[17] Félix Savart (1791 – 1841), French physicist and mathematician

[18] The section of conductor is much smaller than its length, thus current density is constant on the conductor section

The last remaining option would be the magnetic field of a volume current described by volume current density $\vec{J}(\vec{r})$:

$$\vec{H}(\vec{r}) = \int_{(S)} \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{4\pi |\vec{r} - \vec{r}\,'|^3} dV(\vec{r}\,') \quad \text{(Volume Current)} \tag{3.4}$$

By doing similar experiments as Biot and Savart, Ampère discovered another important law that is a special case of the Biot-Savart Law that can be derived from (3.1). Ampère's law says that the line integral of the magnetic field vector over any closed loop is equal to the sum of the currents flowing through any surface bordered by that loop. Mathematically written this looks as follows:

$$\oint_{(L)} \vec{H}(\vec{r}\,') \cdot d\vec{l}(\vec{r}\,') = \sum_{i=1}^{N} I_i \tag{3.5}$$

The situation related to Ampere's Law (3.5) is illustrated in the Figure 3.1.

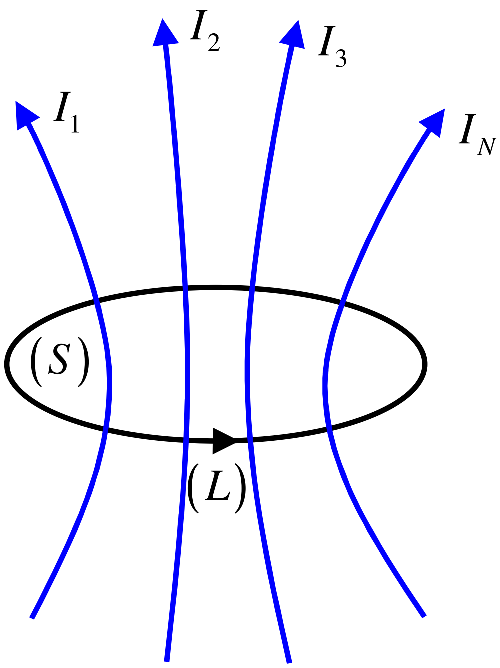

Figure 3.1. Ampère's law; L - closed integration curve; S - surface bordered by L;
$I_1, I_2, \ldots, I_N$ - line electric currents flowing through the surface S

The sum of currents is actually equal to the total current through the surface bordered by the integration loop. This can be written using volume current density $\vec{J}(\vec{r})$ as follows:

$$\oint_{(L)} \vec{H}(\vec{r}\,') \cdot d\vec{l}(\vec{r}\,') = \iint_{(S)} \vec{J}(\vec{r}\,') \cdot \vec{n}(\vec{r}\,')\, dS(\vec{r}\,') \tag{3.6}$$

If we apply Stoke's theorem (curl – theorem) [1, 2] on the left-hand side of (3.6) we obtain the following:

$$\iint_{(S)} \nabla \times \vec{H}(\vec{r}\,') \cdot \vec{n}(\vec{r}\,')\, dS(\vec{r}\,') = \iint_{(S)} \vec{J}(\vec{r}\,') \cdot \vec{n}(\vec{r}\,')\, dS(\vec{r}\,') \tag{3.7}$$

Since the integration area (S) in equation (3.7) is a surface of arbitrary shape, the vector functions under the integrals on left- and right-hand side must be equal. Thus the differential form of Ampère's law has the following form:

$$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) \tag{3.8}$$

3.3. Magnetic Flux Density; Source-Free Character of Magnetostatic Field

The magnetic flux density $\vec{B}$ is introduced here for a similar reason as the electric flux density $\vec{D}$ was introduced in the case of the electric field, represented by equation (1.12). In mathematical form this looks as follows:

$$\vec{B}(\vec{r}) = \mu_0 \cdot \vec{H}(\vec{r}) \tag{3.9}$$

where μ_0 is a constant called *the magnetic permeability of free space* describing the magnetic properties of free space. Using SI units, the permeability of a vacuum has the following value:

$$\mu_0 = 4 \cdot \pi \cdot 10^{-7} \left(H / m\right) \tag{3.10}$$

The abbreviation of unit $\left(H / m\right)$ comes from Henry per meter.

One of the fundamental laws of electromagnetic theory is directly related to the magnetic flux density, i.e. to its flux through any closed surface. Namely, up to now an isolated magnetic pole has not been observed.

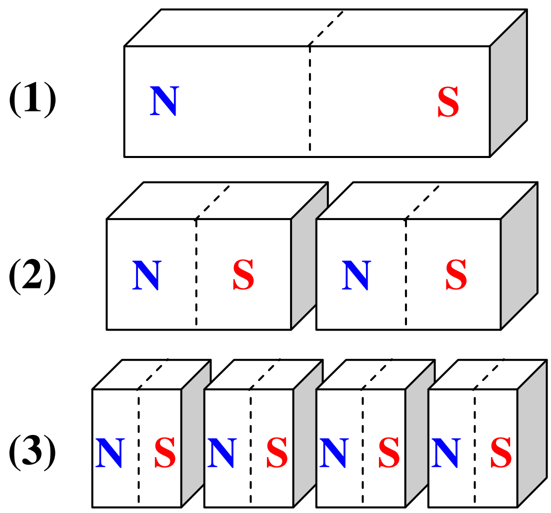

Figure 3.2. A simple illustration of the source-free character of magnetic field;
(1) permanent magnet example with two magnetic poles (N – north pole, S- south pole);
(2) by cutting the original magnet its poles can not be separated, but two new complete magnets with both poles are created;
(3) the cutting procedure will always produce more and more complete magnets but never a single isolated magnetic pole

In a mathematical sense the absence of free magnetic poles can be described as follows:

$$\oint\!\!\!\oint_{(S)} \vec{B}(\vec{r}) \cdot d\vec{S}(\vec{r}) = 0 \qquad (3.11)$$

The situation becomes clearer if one takes into consideration the example depicted in Figure 4. If we have a look at a permanent magnet, it always has two opposite magnetic poles. If one wants to separate or isolate a single magnetic pole (either the north- or the south-pole) by cutting or breaking it along its line of symmetry, it will not be possible. It is simply not possible and it has been never observed in nature, so one can be confident that this law is correct. By applying the divergence theorem to equation (3.11), the differential form of this law is obtained:

$$\nabla \cdot \vec{B}(\vec{r}) = 0 \qquad (3.12)$$

The situation with the integral (3.11) is different if we consider an open surface. Thus we can define the so-called magnetic flux:

$$\Phi = \iint_{(S)} \vec{B}(\vec{r}) \cdot d\vec{S}(\vec{r}) \qquad (3.12')$$

As we will see later the magnetic flux will play an important role in the basic laws of electrodynamics. The SI unit for the magnetic flux is the Weber (the Weber is the Tesla multiplied by the square meter, i.e. $Wb = Tm^2$) and it is named after W. E. Weber[19].

3.4. Magnetic Force

It was already presented in section 1.2 how electric force acts on an electric charge according to Coulomb's Law. Besides the electrostatic force it has been observed that under specific conditions (existing magnetostatic field) an additional force on an electric charge appears due to its movement with finite velocity. Hence, the electromagnetic, or Lorentz[20] [3-5] force acting on a very small (as small as the mathematical point) electric charge in empty space (vacuum) can be interpreted as a visible manifestation of the existence of both the electric and magnetic fields. This force is defined by the following vector equation:

$$\vec{F}_L = q\vec{E} + q\,\vec{v} \times \vec{B} \qquad (3.13)$$

where:

- q – very small electric charge (small in the sense of size – geometry, as well as in the sense of the amount of carrying electric charge)
- $\vec{E}$ - electric field
- $\vec{B}$ - magnetic induction
- $\vec{v}$ - speed of charge

Careful analysis of equation (3.13) shows that the first component of force (related to the electric field $\vec{E}$) exists always when the charge exists and the second component (related to

[19] Wilhelm Eduard Weber (1804 – 1891), German physicist
[20] Hendrik Antoon Lorentz (1853-1928), Dutch theoretical physicist

magnetic induction $\vec{B}$) depends not only on the charge but also on the movement of the point charge, i.e. on its velocity.

The equation looks very simple, but only at first sight. There are several problems hidden in it. For example, the electric charge q is a “source” of its own static electric field in the space around the body. This local field of our trial charge will disturb the field previously existing in the charge-free space. So, by measuring the force (3.13) we always measure the effect of superposition of the initial field in the charge-free space and the local field of the charge itself that is used as a probe. To overcome this problem (or to reduce the error as much as possible) the trial charge has to be very small in the geometrical sense (its volume is so small that tends to be a point, i.e. point charge) as well as in electrical sense (charge of the probe tends to be zero). Using mathematical language, the later statement can be described by the following equation:

$$\lim_{q\to 0}\left(\frac{\vec{F}_L}{q}\right)=\vec{E}+\vec{v}\times\vec{B} \quad (3.14)$$

In order to define, for example, only the electric field one can use equation (3.14) and set the velocity of the trial charge to zero ($v = 0$). It would lead us to the following equation:

$$\vec{E}=\lim_{q\to 0}\left(\frac{\vec{F}_L}{q}\right),\ \vec{v}=0 \quad (3.15)$$

By knowing the electric field (3.15), one can accelerate the charge to a velocity $\vec{v}$ and, while keeping the velocity constant (stationary condition), the force (3.14) can be measured. Comparing this result with the result for a motionless trial charge, one can determine the magnetic induction $\vec{B}$. According to the International System of Units (SI[21]) the unit for electric field is the Newton[22] per Coulomb[23] (N/C), or more frequently the Volt[24] per meter (V/m). The unit of magnetic induction is the Tesla[25] (T).

3.5. Magnetic Scalar and Vector Potentials

It was already shown in the electrostatic case that one can simplify the field computation by introducing a scalar potential as in (1.15). A similar approach can be used in magnetostatics as well. The scalar magnetic potential can be defined as follows:

$$\vec{H}(\vec{r})=-\nabla\cdot V_m(\vec{r}),\ \text{when}\ \vec{J}(\vec{r})=0 \quad (3.16)$$

The situation here is a bit more complicated compared to electrostatics. Namely, due to equation (3.8) the curl operator of a magnetic field is not equal to zero in magnetostatics but is equal to current density. Hence, the magnetic scalar potential can be defined only in the absence of current density.

The partial differential equation of scalar magnetic potential can be derived using (3.16) and (3.12) in the following way:

[21] Le Système international d'unités (SI), Le Bureau international des poids et mesures (BIPM), Paris, France
[22] Isaac Newton (1643-1727), British physicist and mathematician
[23] Charles Auguste de Coulomb (1763-1806), French physicist
[24] Alessandro Giuseppe Antonio Anastasio Volta (1745-1827), Italian physicist
[25] Nikola Tesla (1856-1943), Serbian inventor, physicist and engineer

$$\nabla \cdot \left[\mu_0 \cdot \vec{H}(\vec{r}) \right] = \nabla \cdot \left[-\mu_0 \cdot \nabla V_m(\vec{r}) \right] = 0 \;\Rightarrow\; \boxed{\Delta V_m(\vec{r}) = 0}, \text{ when } \vec{J}(\vec{r}) = 0 \tag{3.17}$$

Obviously, the scalar magnetic potential in a certain domain satisfies the Laplace equation.
In the situation when the current density at a certain point or in a certain domain is not equal to zero one can introduce the vector magnetic potential:

$$\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r}) \tag{3.18}$$

The differential equation of magnetic vector potential can be derived using (3.8) and (3.12) as follows:

$$\nabla \times \vec{H}(\vec{r}) \overset{(2.9)}{=} \nabla \times \left[\frac{1}{\mu_0} \vec{B}(\vec{r}) \right] \overset{(2.15)}{=} \nabla \times \left[\frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r}) \right] \overset{[28]}{=} \frac{1}{\mu_0} \left[\nabla \left(\nabla \cdot \vec{A}(\vec{r}) \right) - \Delta \vec{A}(\vec{r}) \right] \overset{(2.15)}{=} \vec{J}(\vec{r}) \tag{3.19}$$

At this point we need to realize that by simply stating equation (3.18) the magnetic vector potential is not fully defined. In order to uniquely define it one must know *both* spatial derivatives ($\nabla \cdot$, $\nabla \times$) of the function in the same time[26]. The curl operator of magnetic vector potential is defined by (3.18). The vector potential divergence definition is called *gauging*, as in [6]. For magnetostatics a zero divergence of a magnetic vector potential appears to be the most useful choice and this is called the Coulomb gauge:

$$\nabla \cdot \vec{A}(\vec{r}) = 0 \tag{3.20}$$

Having gauge (3.20) equation (3.19) becomes the differential equation of magnetic vector potential:

$$\boxed{\Delta \vec{A}(\vec{r}) = -\mu_0 \cdot \vec{J}(\vec{r})} \tag{3.21}$$

If we recall briefly the knowledge of electrostatics, it is obvious that equation (3.18) is very similar to the Poisson equation of electrostatic potential (1.45). The only difference is that here we have a vector equation. Furthermore, in electrostatics the solutions of Poisson's equation in integral form represented by (1.17 – 1.19) are well known. Following the same logic one can now write the integral form of the solution of (3.21) as follows:

$$\vec{A}(\vec{r}) = \iiint\limits_{(V)} \frac{\mu_0 \vec{J}(\vec{r}\,')}{4\pi |\vec{r} - \vec{r}\,'|} dV(\vec{r}\,'), \text{ for volume current density} \tag{3.22}$$

$$\vec{A}(\vec{r}) = \iint\limits_{(S)} \frac{\mu_0 \vec{K}(\vec{r}\,')}{4\pi |\vec{r} - \vec{r}\,'|} dS(\vec{r}\,'), \text{ for surface current density} \tag{3.23}$$

$$\vec{A}(\vec{r}) = \int\limits_{(L)} \frac{\mu_0 I}{4\pi |\vec{r} - \vec{r}\,'|} d\vec{l}(\vec{r}\,'), \text{ for line current} \tag{3.24}$$

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[26] Helmholtz's theorem.

3.6. Magnetostatic field in Materials; Magnetisation

In Chapter 1.4 we analyzed the reaction of a material to an external electric field. A similar analysis also holds for magnetic field. As we have shown in equation (3.13), the component of the Lorentz force related to magnetic field acts on a charge when it moves. We already gave a simple picture of atoms where the movement of electrons around the nucleus permanently exists. Moreover, every electron in the atom shells rotates around its own axis (spin of electron), which appears to be even more significant from the external magnetic field and magnetic behaviour of materials viewpoints [6]. Similar to the previous analysis of the electric polarisation of dipoles it is now logical to introduce *the magnetic dipole* and *magnetic dipole moment* as shown in the Figure 5.
The simplified description of atom assumes that it is usually a point charge orbiting around a centre that forms a current loop and consequently a magnetic dipole. The basic physical value that determines the magnetic dipole is its magnetic moment. The magnetic moment of the circular current loop is defined as:

$$\vec{m} = \vec{n} \cdot i \cdot S \tag{3.25}$$

where $\vec{n}$ is the unit vector perpendicular to the surface of loop, i is the electric current of the loop and S is the area of the loop. The SI unit for the magnetic moment of a dipole is the Ampère multiplied by square meter (Am^2). Having definition (3.25), it is not difficult to derive the magnetic moment of the dipole depicted in the Figure 4 – b:

$$\vec{m} = \vec{n} \cdot i \cdot S = \vec{n} \cdot \frac{\partial Q}{\partial t} \cdot R^2 \pi = \vec{n} \cdot \frac{\partial}{\partial t}\left(\frac{v \cdot t}{2R\pi} q\right) \cdot R^2 \pi = \vec{n} \cdot \frac{v \cdot R}{2} q = \frac{\vec{R} \times \vec{v}}{2} q$$

$$\vec{m} = \frac{\vec{R} \times \vec{v}}{2} q \tag{3.26}$$

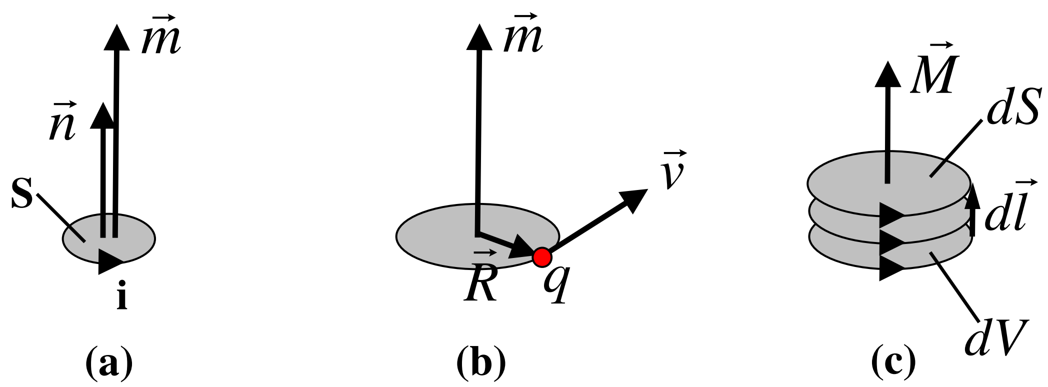

Figure 3.3. The magnetic dipole is depicted; (a) a small current loop as a magnetic dipole; (b) a point charge orbiting around a centre as a magnetic dipole; (c) a macroscopic representation of the magnetic dipoles in a certain material.

Definition (3.26) corresponds to the microscopic behaviour of a certain material. In order to get to the level of macroscopic field theory it is necessary to define the *magnetisation vector* or the *magnetic moment density*:

$$\vec{M}(\vec{r}) = \lim_{\Delta V \to 0} \frac{\sum_i \vec{m}_i}{\Delta V} = \frac{\partial \vec{m}}{\partial V} \tag{3.27}$$

The magnetisation vector describes the magnetic properties of a material exposed to an external magnetic field. The atoms or molecules of the material can be represented as magnetic dipoles. In a macroscopic sense the dipoles of a material can be described by so called Ampère's currents flowing inside the material and by the magnetisation vector. Without an external magnetic field the Ampère's currents are chaotically distributed in space (according to the molecular kinetic theory). After the appearance of an external magnetic field the internal magnetic dipoles of the material start to have a less chaotic distribution and tend to orient their axes towards the direction of the external field. This dramatically changes the magnetic situation inside the material. In order to derive the connection between Ampère currents and magnetisation, let us consider the situation presented in Figure 4-c. A very small volume dV contains a certain amount of Ampère currents. According to the definition of the magnetisation vector (3.27) it is possible to write:

$$d\vec{m}(\vec{r}) = \vec{M}(\vec{r}) \cdot dV \overset{(Figure4-c)}{=} \left(\vec{M}(\vec{r}) \cdot d\vec{l}\right) \cdot \vec{n}_S \cdot dS \tag{3.28}$$

By comparing equation (3.25) and equation (3.28) the following is obvious:

$$\vec{M}(\vec{r}) \cdot d\vec{l} = dI_a \tag{3.29}$$

where dI_a represent the Ampère currents associated with the volume dV. In order to compute the total Ampère current through a surface it is necessary to integrate the Ampère current density over the surface or, according to (3.29), the magnetisation vector over the boundary curve of the surface:

$$I_a = \iint_{(S)} \vec{J}_a(\vec{r}) \cdot d\vec{S}(\vec{r}) \overset{(53)}{=} \oint_{(\Gamma)} \vec{M}(\vec{r}) \cdot d\vec{l}(\vec{r}) \tag{3.30}$$

Using (3.30) and Stoke's theorem (curl – theorem) [1, 2] it is not difficult to see the following:

$$\vec{J}_a(\vec{r}) = \nabla \times M(\vec{r}) \tag{3.31}$$

Since the nature of the Ampère currents is purely solenoidal, i.e. it is obtained as a result of the curl-operator in (3.31), the divergence of the Ampère current density must be equal to zero:

$$\nabla \cdot \vec{J}_a(\vec{r}) = 0 \tag{3.32}$$

Having equation (3.31) and using equation (3.8) (Ampere's Law) one can draw an important conclusion:

$$\nabla \times \vec{B}(\vec{r}) \overset{(2.8)}{=} \mu_0 \vec{J}(\vec{r}) + \mu_0 \vec{J}_a(\vec{r}) \overset{(2.31)}{=} \mu_0 \vec{J}(\vec{r}) + \mu_0 \nabla \times \vec{M}(\vec{r}) \tag{3.33}$$

$$\nabla \times \left[\frac{1}{\mu_0} \vec{B}(\vec{r}) - \vec{M}(\vec{r}) \right] = \vec{J}(\vec{r}) \tag{3.34}$$

It is now convenient to define a new vector as a representation of the terms under the curl-operator on the left-hand side of (3.34):

$$\vec{H}(\vec{r}) = \frac{1}{\mu_0} \vec{B}(\vec{r}) - \vec{M}(\vec{r}) \tag{3.35}$$

The vector $\vec{H}$ is called the *magnetic field* . Compared to definition (3.9), which is valid only for free space, equation (3.35) is valid for any kind of media. By introducing it into equation (3.34) we obtain the following:

$$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) \tag{3.36}$$

For an isotropic media the magnetic polarisation vector is parallel to the magnetic field with a certain coefficient of proportionality:

$$\vec{M}(\vec{r}) = \chi_m \vec{H}(\vec{r}) \tag{3.37}$$

where χ_m is called *magnetic susceptibility*. Using (3.35) it is possible to write:

$$\vec{H}(\vec{r}) = \frac{1}{\mu_0} \vec{B}(\vec{r}) - \chi_m \vec{H}(\vec{r}) \quad \Rightarrow \quad \vec{H}(\vec{r}) = \frac{\vec{B}(\vec{r})}{\mu_0 (1 + \chi_m)} \tag{3.38}$$

The constant of proportionality between the magnetic field $\vec{H}$ and the magnetic induction $\vec{B}$ is usually written in a more appropriate form:

$$\mu_0 (1 + \chi_m) = \mu_0 \mu_r = \mu \tag{3.39}$$

where μ_r is dimensionless and called the *relative magnetic permeability* of a material, and μ is the *magnetic permeability* of a material. The SI unit for magnetic permeability is the Tesla multiplied by meter over Ampere (Tm/A).
In some real-life situations the magnetic permeability depends on the magnetic field, i.e. $\mu = \mu(H)$ and those materials are called magnetically nonlinear materials.

3.7. Laplace's and Poisson's Equations

In a similar way as in electrostatics (see Section 1.5) we will now derive the basic partial differential equations of magnetostatic fields. To make our approach general, i.e. valid in the case of an existing volume current density in certain parts of a computational domain, we will use here the vector magnetic potential as in (3.18). Our analysis will start with equations (3.38) and (3.39) written in a compact form:

$$\vec{B}(\vec{r}) = \mu(\vec{r}) \, \vec{H}(\vec{r}) \tag{3.40}$$

where the magnetic permeability $\mu(\vec{r})$ is considered a spatial function, i.e. our material is considered inhomogeneous. If we divide equation (3.40) with $\mu(\vec{r})$ and include (3.18), it reads:

$$\frac{1}{\mu(\vec{r})}\nabla\times\vec{A}(\vec{r})=\vec{H}(\vec{r}) \tag{3.41}$$

By applying the curl operator ($\nabla\times$) on equation (3.41) and inserting equation (3.36) into it, we obtain the following very important vectorial partial differential equation (PDE) of magnetostatic field:

$$\nabla\times\left[\frac{1}{\mu(\vec{r})}\nabla\times\vec{A}(\vec{r})\right]=\vec{J}(\vec{r}) \tag{3.42}$$

Depending on the magnetic permeability, i.e. the magnetic properties of a material, equation (3.42) can be reduced to a simpler form. Thus for a homogeneous material ($\mu\neq\mu(\vec{r})$), utilizing the following well known vector identity [7, 8]:

$$\nabla\times\left(\nabla\times\vec{a}\right)=\nabla\left(\nabla\cdot\vec{a}\right)-\Delta\vec{a} \tag{3.43}$$

and including gauge (3.20) we obtain the vectorial Poisson's equation for the magnetic vector potential:

$$\Delta\vec{A}(\vec{r})=-\mu\,\vec{J}(\vec{r}) \tag{3.44}$$

When the volume current density is equal to zero equation (3.44) reduces to the vectorial Laplace's equation for the magnetic vector potential:

$$\Delta\vec{A}(\vec{r})=0 \tag{3.45}$$

Equation (3.44) is the one we usually solve when computing a magnetostatic field with an existing volume current density. Obviously, it consists of three scalar component equations that have to be solved simultaneously. When the volume current density is absent, instead of solving (3.45), equation (3.17) is usually used. Thus, the problem of magnetostatic field computation can be reduced to the scalar Laplace's equation.

3.8. Magnetostatic Boundary Value Problem (BVP)

Since we have already seen the structure of the electrostatic BVP (see Section 1.6), it is not necessary to repeat the theoretical statements on PDEs and their associated boundary conditions. Thus we will present only the partial equation and boundary conditions characteristic for magnetostatics. The usual formulation of the magnetostatic BVP that is widely accepted has the following form:

$$\nabla\times\left[\frac{1}{\mu(\vec{r})}\nabla\times\vec{A}(\vec{r})\right]=\vec{J}(\vec{r})\,,\ \vec{r}\in\Omega\subseteq R^3 \tag{3.46}$$

$$\vec{A}(\vec{r}) = 0,\ \vec{r} \in \partial\Omega_{MI} \tag{3.47}$$

$$\vec{n} \times \vec{H}(\vec{r}) = 0,\ \vec{r} \in \partial\Omega_{EI} \tag{3.48}$$

Boundary condition (3.47) is called the magnetic insulation because it sets the magnetic vector potential and therefore its normal component over the boundary ($\partial\Omega_{MI}$) to zero. This *magnetically insulates* the domain bordered by this surface from the surrounding space. On the other hand boundary condition (3.48) sets the tangential component of the magnetic field equal to zero over a certain boundary. This implies that the normal component of the volume current density over the surface ($\partial\Omega_{EI}$) is equal to zero, which *electrically insulates* our domain from the remaining space.
Similar to the previous electrostatic case (Section 1.6), one can solve the magnetostatic BVP (3.46 -3.48) using an analytical (simple and symmetric geometry) or numerical (general) approach.

3.9. Force and Energy in Magnetostatics

In Section 3.4 the magnetic force on a point charge moving with velocity $\vec{v}$ in a magnetic field with induction $\vec{B}$ was qualitatively and quantitatively described. We will use this knowledge now to derive the force on the current carrying conductor of an arbitrary shape with a small cross section compared to its length when subjected to an external magnetic field. This situation is presented in Figure 3.4.

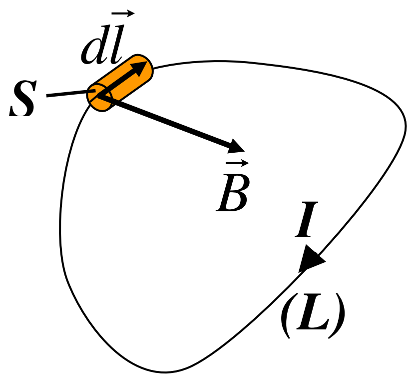

Figure 3.4. A current carrying conductor loop is depicted (L); The conductor loop is exposed to the influence of an external magnetic field with magnetic induction ($\vec{B}$); It is assumed that the conductor has a cross section of constant shape (S) which is much smaller than its length.

The electric current flow in the conductor is essentially the directed movement of the free electrons inside the conductor material. Each of these moving electrons as a charge carrier is affected by the magnetic Lorentz force (3.13):

$$\vec{F}_L = q\,\vec{v} \times \vec{B} \tag{3.49}$$

Therefore, an element dl of the conductor will be affected by the magnetic force that can be calculated using (3.49) as follows:

$$d\vec{F} = \rho\ S\ dl\ \vec{v} \times \vec{B} \tag{3.50}$$

where S is the cross sectional area of the conductor, ρ is the charge density, $\vec{v}$ is the average speed of the electrons and $\vec{B}$ is the external magnetic induction. Since $\vec{J} = \rho\vec{v}$ (see equation (2.5)) is the volume current density and $I = JS$ (see equation (2.6)) is the intensity of the electric current, the force (3.50) reads:

$$d\vec{F} = I\, d\vec{l} \times \vec{B} \tag{3.51}$$

Therefore the total force acting on the loop (L) can be calculated as:

$$\vec{F} = \oint_{(L)} I\, d\vec{l} \times \vec{B} \tag{3.52}$$

The energy of a magnetostatic field can be elegantly derived starting only from the dynamic field equations. Therefore, it is perhaps useful to first write the final result, than to start with the derivations at the electrodynamic stage. Thus, the energy stored in a magnetostatic field in a linear medium can be written as follows [3]:

$$W_m = \frac{1}{2}\iiint_{(V)} \vec{B}\cdot\vec{H}\; dV = \frac{1}{2}\iiint_{(V)} \mu H^2\; dV = \frac{1}{2}\iiint_{(V)} \frac{B^2}{\mu}\; dV \tag{3.53}$$

When the magnetic material is nonlinear the energy reads:

$$W_M = \int_0^t d_t W_M = \iiint_{(V)} \left\{ \int_0^t \vec{H}(\vec{r})\cdot d_t\vec{B}(\vec{r}) \right\} \cdot dV(\vec{r}) \tag{3.54}$$

It is worth mentioning that equation (3.54) is very similar to equation (1.61) derived in Section 1.8 for electrostatic field.

3.10. Interface Conditions

Using Ampère's Law (3.5) and the solenoidal character of magnetostatic fields (3.11) expressed in an integral form, one can draw important conclusions about the behaviour of magnetostatic field at the interface between different materials. Let us consider the situation depicted in Figure 3.5.

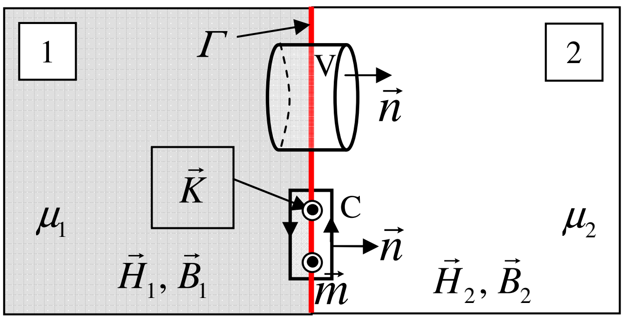

Figure 3.5. Interface between two different materials (materials are different in magnetic sense); V – small cylindrical volume; C – small contour; σ - surface density of electric charge over interface; Γ - surface boundary (interface) between two domains.

Since we have done a similar analysis in Section 1.9 on electrostatic fields we will not now repeat the detailed derivation. Thus, using the solenoidal character of magnetostatic fields (3.11) for the volume (V) in Figure 3.5 it is possible to write the following:

$$\oiint_{(\partial V)} \vec{B}(\vec{r}) \cdot \vec{n}(\vec{r})\, dS(\vec{r}) = 0 \tag{3.55}$$

By following the same procedure as described in the equations (1.69-1.71) an important conclusion is obtained:

$$\boxed{\left(\vec{B}_1 - \vec{B}_2\right) \cdot \vec{n} = 0} \tag{3.56}$$

Then applying the Ampère's Law to the contour (C) in Figure (3.5) one obtains:

$$\oint_{(C)} \vec{H}(\vec{r}) \cdot d\vec{l}(\vec{r}) = \iint_{(S)} \vec{J}(\vec{r}) \cdot d\vec{S}(\vec{r}) \tag{3.57}$$

Similar to the derivation of equations (1.73-1.75), equation (3.57) leads us to the following conclusion:

$$\boxed{\left(\vec{H}_1 - \vec{H}_2\right) \times \vec{n}(\vec{r}) = \vec{K}} \tag{3.58}$$

where $\vec{K}$ is the surface current flowing over the surface interface. Equations (3.56) and (3.58) are important interface conditions for magnetostatic field that have to be fulfilled over the boundary between two materials with different magnetic properties. As we will see later, these conditions are very important for magnetostatic field computations.

3.11. References

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