5. Field Discontinuities, S-parameters and Lumped Element Representation

5.1. Introduction

Modern electronic devices are very complex structures consisting of thousands or even millions compact elements connected to each other. Electromagnetic simulation of such devices involving all components is possible from theoretical point of view, but practically it is not attainable. For the analysis of such complex systems it is much more convenient to apply the so-called *circuit theory*. In this theory each component is considered as a black box described by the *lumped electromagnetic parameters* obtained as electromagnetic field integrals over the volume of the component or element [1]. Those lumped elements are connected to each other with ideal electric wires[1] thus forming an *electric network*. A typical graphical representation of a simple electric network is given in Figure 5.1.

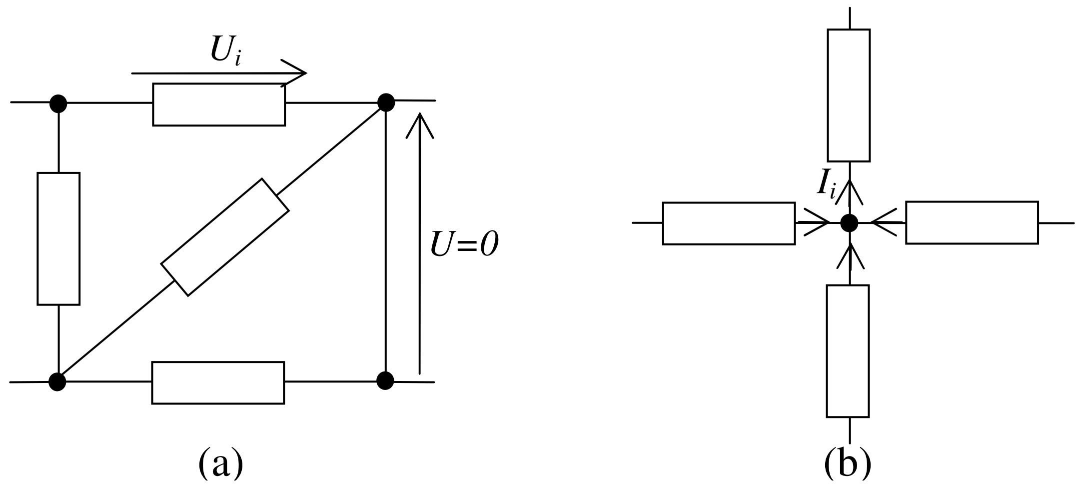


Figure 5.1. A simple electric network is presented (a). It is important to see that the voltage between two point of the same wire is zero while the voltage between two different wires connected to the lumped element is in general not equal to zero; When two or more wires are connected the so-called node is obtained (b).

Since electric wires are made of ideal conductor it is assumed that the wire can not carry any charge and the voltage along them has a constant value (a voltage difference between any two points on the same wire is equal to zero). Two or more electric wires connected to each other form a node that is also assumed to be an ideal object and therefore can not carry any charge [1].

Generally speaking, in the electric circuits theory we deal only with the integral parameters such as lumped elements, voltages and currents. Those parameters are apparently integral representations of the existing electromagnetic field in the structure. In the next section will be therefore given the fundamental field laws in the form appropriate for the electric circuits theory.

5.2. Kirchhoff's Laws

The basic laws of the circuits theory can be derived starting from the fundamental equations for the stationary current distribution presented in Chapter 2. Namely, we will use here two equations given in Table 2.1 describing the divergence of the volume current density and the curl of the electric field:

[1] Ideal wire has no losses and the current flows only inside of the wire assuming that surrounding space is an ideal dielectric with its conductivity equal to zero.

$$\nabla \cdot \vec{J}(\vec{r}) = 0 \tag{5.1}$$

$$\nabla \times \vec{E} = 0 \tag{5.2}$$

Let us integrate the equation (5.1) over the volume of a single element in the network and the equation (5.2) along the surface of one closed loop in the network. Thus we obtain the following equations:

$$\iiint_{(V)} \nabla \cdot \vec{J}(\vec{r})\, dV = 0 \tag{5.3}$$

$$\iint_{(S)} \nabla \times \vec{E} \cdot d\vec{S} = 0 \tag{5.4}$$

Applying the theorem of Gauss [2, 3] on (5.3) and the theorem of Stokes [4, 5] on (5.4) those equations become:

$$\oiint_{(\partial V)} \vec{J}(\vec{r}) \cdot d\vec{S} = 0 \tag{5.5}$$

$$\oint_{(\partial S)} \vec{E} \cdot d\vec{l} = 0 \tag{5.6}$$

As it has been presented in Section 2 by the equation (2.6), the electric current through a certain surface can be calculated as:

$$I = \iint_{(\partial V)} \vec{J}(\vec{r}) \cdot d\vec{S} \tag{5.7}$$

By following the equation (1.21) from a Chapter on electrostatics and by taking into the consideration Table 2.1 describing the analogy between the electrostatic field and the stationary current distribution on can write the equation for a voltage between two wires following the path *(l)* as follows:

$$V = \int_{(l)} \vec{E} \cdot d\vec{l} \tag{5.8}$$

Having the expressions for the voltage (5.8) and current (5.7) one can write the integral laws of field theory (5.5) and (5.6) in the circuits theory form:

$$\oiint_{(\partial V)} \vec{J}(\vec{r}) \cdot d\vec{S} = \sum_{i=1}^{N} I_i = 0 \tag{5.9}$$

$$\oint_{(\partial S)} \vec{E} \cdot d\vec{l} = \sum_{i=1}^{N} U_i = 0 \tag{5.10}$$

The equation (5.9) represents the first Kirchhoff's law saying that the algebraic sum of all currents coming to the same is equal to zero. The equation (5.10) is the second Kirchhoff's law showing that the algebraic sum of all voltages along any closed loop of the network is equal to zero. It is worth mentioning that those laws are derived starting from the equations of the stationary current distribution and as such are valid for DC currents. Note that more accurate laws valid for time varying currents can be obtained from non-static Maxwell's equations [1].

5.3. Capacitance, Inductance and Resistance

In this section we will define the most simple lumped elements that have only two wires. Let us imagine two conductive electrodes close to each other. If we put those electrodes at different potentials (constant in time), the electrostatic charge will distribute over the surface of the electrodes and electrostatic field will appear in the space between them. As we know from Chapter 1 the electrostatic field inside of ideal conductor is zero. This system of two electrodes is called *capacitor.* It is assumed that the total charge of the capacitor is equal to zero. Thus two electrodes have opposite charge +Q and –Q. The lumped parameter of the capacitor is the *capacitance* [1]:

$$C = \frac{Q}{U} = \frac{\iint\limits_{(S_e)} \vec{D} \cdot d\vec{S}}{\int\limits_{(l)} \vec{E} \cdot d\vec{l}} \tag{5.11}$$

where Q is the total charge of one electrode, U is the voltage (potential difference) between the electrodes, S_e is the surface of the electrode, and l is an arbitrary path between the electrodes. Apparently the capacitance is determined by the electrostatic field of the capacitor. The symbol for electric capacitance used in electric schemes is shown in Figure 5.2a. The SI unit for the capacitance is the *farad* (the farad is the coulomb over the volt, i.e. $F = C/V$) and it is named after Michael Faraday.

Similar to the previous case we can have a lumped element with magnetostatic field inside instead of electrostatic field. The lumped parameter of such magnetostatic element is called *inductor* and it is represented by the *inductance* [1]:

$$L = \frac{\Phi}{I} = \frac{\iint\limits_{(S_m)} \vec{B} \cdot d\vec{S}}{\iint\limits_{(S)} \vec{J} \cdot d\vec{S}} = \frac{\iint\limits_{(S_m)} \vec{B} \cdot d\vec{S}}{\oint\limits_{(\partial S)} \vec{H} \cdot d\vec{l}} \tag{5.12}$$

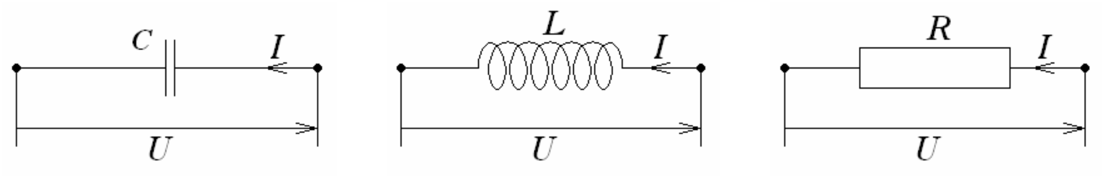
where Φ is the magnetic flux, I is the corresponding electric current (that produces the magnetic flux), S_m is the surface breached by the magnetic flux, S is the surface flowed by the electric current. The symbolic representation of the inductance is given in Figure 5.2b. The SI unit for the inductance is the *henry* (the henry is the weber over the ampere, i.e. $H = Wb/A$) in honour of J. Henry[2].

At the end, one can have a lumped element with the static current field inside is called the *resistor* and it is represented by the *resistance* [1]:

[2] Joseph Henry (1797 – 1878), Scottish-American physicist

$$R = \frac{U}{I} = \frac{\int\limits_{(l)} \vec{E} \cdot d\vec{l}}{\iint\limits_{(S)} \vec{J} \cdot d\vec{S}} \tag{5.13}$$

where U is the voltage connected to the contacts of the resistor, I is the current through the resistor, l is the path from one contact of the resistor to the other contact, and S is any cross section of the resistor. The symbolic representation of the inductance is given in Figure 5.2c. The SI unit for the resistance is the *ohm* (the ohm is the volt over the ampere, i.e. $\Omega = V/A$) named after G. Ohm[3].



(a) (b) (c)

Figure 5.2. Symbolic representation of capacitor (a), inductor (b) and resistor (c); Presented symbols are used in schematic diagrams of the circuit theory representing the electric networks.

To use this lumped elements in circuit theory, the relation between the voltage at the contacts of the element and the electric current through the element is needed. Therefore the electric charge Q in (5.11) and the magnetic flux Φ in (5.12) have to be eliminated. This is an easy task if we apply the following relations:

$$I(t) = \frac{\partial Q}{\partial t}(t) \tag{5.14}$$

$$U(t) = \frac{\partial \Phi}{\partial t}(t) \tag{5.15}$$

The equation (5.14) has been already mentioned in Chapter 2 on stationary current distribution (see the equations (2.2) and (2.3)). The equation (5.15) represents the induced voltage generated by the time varying magnetic field and it is derived from Faraday's law presented in Chapter 4 on electrodynamics (see the equation (4.6)). Having (5.14) and (5.15) we can write the following:

$$(5.11) \Rightarrow Q(t) = C \cdot U(t) \quad \Big| \frac{\partial}{\partial t} \quad \overset{(5.14)}{\Rightarrow} \quad \boxed{I(t) = C \cdot \frac{\partial U}{\partial t}(t)} \tag{5.16}$$

$$(5.12) \Rightarrow \Phi(t) = L \cdot I(t) \quad \Big| \frac{\partial}{\partial t} \quad \overset{(5.15)}{\Rightarrow} \quad \boxed{U(t) = L \cdot \frac{\partial I}{\partial t}(t)} \tag{5.17}$$

The electric resistor, capacitor and inductor represented by the equations (5.13), (5.16) and (5.17) respectively are ready for use in the schematic diagrams and equations of circuit theory.

[3] Georg Simon Ohm, (1789 - 1854), German physicist

For harmonic time varying electromagnetic fields we will apply the complex approach already described in Section 4.6 (see the equations (4.34) and (4.35)). Thus, the equations of our basic lumped elements can be in complex (frequency) domain written as follows:

$$\underline{U} = R \cdot \underline{I} \tag{5.18}$$

$$\underline{I} = j\omega C \cdot \underline{U} \tag{5.19}$$

$$\underline{U} = j\omega L \cdot \underline{I} \tag{5.20}$$

where $\underline{U}$ and $\underline{I}$ are the complex representations of the voltage and current of a certain lumped element respectively.
Very often instead of the *resistance* is in use the so-called *conductance*:

$$G = \frac{1}{R} = \frac{I}{U} \tag{5.21}$$

The SI unit for the resistance is the *siemens* (the siemens is the ampere over the volt, i.e. $S = A/V = 1/\Omega$) named after E. W. Siemens[4].
Due to the fact that the capacitance and the inductance introduce the phase shift between the voltage and current of $-\pi/2$ and $\pi/2$, they are called the reactance (they power is not an active power but reactive power). It is practically impossible to have an element that has only a pure resistance or a pure capacitance, for example. Therefore, it is pretty common a serial or parallel combination of an ideal resistance and an ideal reactance (capacitance or inductance, or both). Such combination is called the *impedance* and its serial version looks as follows:

$$\underline{Z} = R + jX \tag{5.22}$$

where X is the reactance which can be the capacitive reactance:

$$X = X_C = -\frac{1}{\omega C} \tag{5.23}$$

the inductive reactance:

$$X = X_L = \omega L \tag{5.24}$$

The reciprocal impedance is the *admittance*:

$$\underline{Y} = \frac{1}{\underline{Z}} = G + jB \tag{5.25}$$

where B is called the susceptance. The SI unit for the impedance and reactance is the ohm. On the other hand, the SI unit for the admitance and susceptance is the siemens.

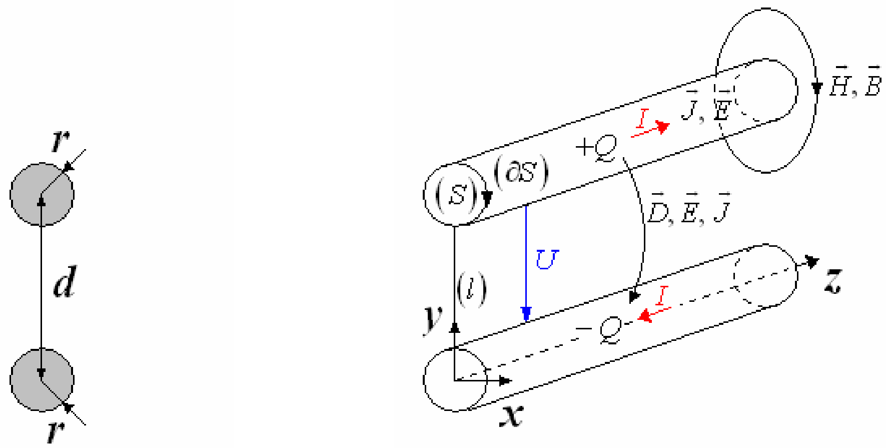
[4] Ernst Werner von Siemens (1816 – 1892), German inventor and industrialist

5.4. Transmission Lines

Transmission lines are systems designed for a transfer of electromagnetic energy between two distant points in the space (usually between the source and the load). They consists of at least two conductive wires surrounded by a certain dielectric material. Although the electromagnetic field of this system is not concentrated (lumped) in a certain small area, it is possible under certain conditions to represent the transmission line with a relatively simple equivalent circuit consisting of capacitors, inductors and resistors. To understand that, let us consider a simple example of two parallel wires with circular cross sections presented in Figure 5.3 and let us assume the following [1]:

- inside of the wires the electric current flows in the z-direction,
- the electric current in the dielectric appears due to the losses and time-varying electromagnetic field and flows in the transversal (x0y) plane.

With these relatively crude assumptions we split the electromagnetic field into two parts: the field inside of the conductors and field in the dielectric.



(a) (b)

Figure 5.3. Cross section of the transmission line consisting of two wires (a); Short section of the transmission line is depicted with the important surfaces and lines for the lumped parameters computation (b); *Q* is the electric charge, *I* is the electric current thorough the wire, *U* is the voltage between the wires.

By using the equation (5.13) and by integrating it over the unit length of the transmission line one can obtain the resistance *R'* per unit length. Similarly, by performing the same integration outside the conductors we obtain the conductance per unit length *G'*. We use the conductance for the dielectric because the conductance element representing the losses in the dielectric will be apparently the parallel branch of the equivalent circuit. In such a situation in circuit theory (parallel branch), it is commonly accepted the rule of using the conductance instead of the resistance. In the same way, by computing the integral in (5.12) and (5.11) one can determine the inductance per unit length *L'* and the capacitance per unit length *C'* of the transmission line. It is worth mentioning that for an accurate computation of *R'* and *L'* the static field analysis can not be used. Namely, due to some pure electrodynamic effects such as the skin and proximity effects one has to perform the electrodynamic field computation for obtaining the accurate results. Thus we obtain the equivalent circuit of the transmission line shown in Figure 5.4.

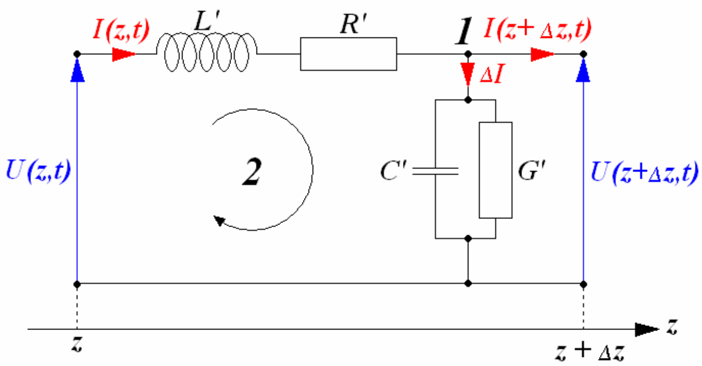


Figure 5.4. The equivalent circuit of the transmission line is shown; L' is the inductance per unit length, C' is the capacitance per unit length, R' is the resistance per unit length, G' is the conductance per unit length; More details can be found in the text..

Let us consider the propagation of the EMW along a transmission line which can be presented by the equivalent circuit shown in Figure 5.4. If we apply the Kirchof's laws (5.9) and (5.10) to the node "1" and loop "2" respectively, we obtain the following equations:

$$U(z,t) \overset{(5.10,5.17,5.18)}{=} R'\Delta z\, I(z,t) + L'\Delta z \frac{\partial I}{\partial t}(z,t) + U(z+\Delta z,t) \quad (5.26)$$

$$I(z,t) \overset{(5.9)}{=} I(z+\Delta z,t) + \Delta I \overset{(5.16,5.18)}{=} I(z+\Delta z,t) + G'\Delta z\, U(z+\Delta z,t) + C'\Delta z \frac{\partial U}{\partial t}(z+\Delta z,t) \quad (5.27)$$

By dividing the equations (5.26) and (5.27) with Δz and moving free terms to the left-hand side we obtain:

$$-\frac{U(z+\Delta z,t)-U(z,t)}{\Delta z} = R' I(z,t) + L'\frac{\partial I}{\partial t}(z,t) \quad (5.28)$$

$$-\frac{I(z+\Delta z,t)-I(z,t)}{\Delta z} = G' U(z+\Delta z,t) + C'\frac{\partial U}{\partial t}(z+\Delta z,t) \quad (5.29)$$

Taking the limit $\Delta z \to 0$ the left-hand sides of (5.28) and (5.29) become the partial derivatives with respect to the variable z:

$$-\frac{\partial U}{\partial z}(z,t) = R' I(z,t) + L'\frac{\partial I}{\partial t}(z,t) \quad (5.30)$$

$$-\frac{\partial I}{\partial z}(z,t) = G' U(z,t) + C'\frac{\partial U}{\partial t}(z,t) \quad (5.31)$$

The equations (5.30) and (5.31) are called the *telegraph equations* in a coupled form [1]. If we want to decouple them it is necessary to perform the derivations of both equations with respect to z and t and than to introduce those derivatives in a corresponding equations. In such a way one can obtain the following decoupled *telegraph equations* [1]:

$$\frac{\partial^2 U}{\partial z^2}(z,t) = \left(R' + L'\frac{\partial}{\partial t}\right)\left(G' + C'\frac{\partial}{\partial t}\right)U(z,t) \quad (5.32)$$

$$\frac{\partial^2 I}{\partial z^2}(z,t)=\left(R'+L'\frac{\partial}{\partial t}\right)\left(G'+C'\frac{\partial}{\partial t}\right)I(z,t) \tag{5.33}$$

For time-harmonic electromagnetic fields the telegraph equations in a frequency domain become [8]:

$$\frac{d^2\underline{U}}{dz^2}(z)=\left(R'+j\omega L'\right)\left(G'+j\omega C'\right)\underline{U}(z) \tag{5.34}$$

$$\frac{\partial^2\underline{I}}{\partial z^2}(z)=\left(R'+j\omega L'\right)\left(G'+j\omega C'\right)\underline{I}(z) \tag{5.35}$$

To write the equations (5.34) and (5.35) in a more compact form it is useful to define the following [8]:

$$\underline{\gamma}=\alpha+j\beta=\sqrt{\left(R'+j\omega L'\right)\left(G'+j\omega C'\right)} \tag{5.35}$$

where γ is the propagation constant, α is the attenuation constant and β is the phase constant [8]. Thus the telegraph equations (5.34) and (5.35) read:

$$\frac{d^2\underline{U}}{dz^2}(z)-\gamma^2\underline{U}(z)=0 \tag{5.36}$$

$$\frac{\partial^2\underline{I}}{\partial z^2}(z)-\gamma^2\underline{I}(z)=0 \tag{5.37}$$

Apparently, we have obtained the same wave equations as in the case of the electromagnetic waves (see the equations (4.43) and (4.44)). The solutions of (5.36) and (5.37) are well known (they are the linear homogenous differential equations):

$$\underline{U}(z)=\underline{U}_0^+\,e^{-\gamma z}+\underline{U}_0^-e^{\gamma z} \tag{5.38}$$

$$\underline{I}(z)=\underline{I}_0^+\,e^{-\gamma z}+\underline{I}_0^-e^{\gamma z} \tag{5.39}$$

where $\underline{U}_0^+$, $\underline{U}_0^-$, $\underline{I}_0^+$, $\underline{I}_0^-$ are arbitrary constants. The meaning of these constants are already explained in Section 4.10 on the plane EMW. Namely, it has been shown that the solution (4.75) of the wave equation (4.74) is the superposition of the progressive and regressive EMW. Therefore the constants $\underline{U}_0^+$, $\underline{U}_0^-$ in (5.38) and (5.39) represent the amplitudes of the progressive and regressive voltage wave respectively. Accordingly, the constants $\underline{I}_0^+$, $\underline{I}_0^-$ in (5.38) and (5.39) represent the amplitudes of the progressive and regressive current wave respectively.
If we calculate the ratio of the progressive voltage wave and corresponding current wave at any point of the line we obtain the *characteristic impedance* of a transmission line:

$$Z_0=\frac{\underline{U}^+(z)}{\underline{I}^+(z)}=\frac{\underline{U}_0^+\,e^{-\gamma z}}{\underline{I}_0^+\,e^{-\gamma z}}=\frac{\underline{U}_0^+}{\underline{I}_0^+}=-\frac{\underline{U}_0^-}{\underline{I}_0^-} \tag{5.40}$$

If we write the equation (5.30) in frequency domain we obtain:

$$-\frac{d\underline{U}}{dz}(z)=\left(R'+j\omega L'\right)\underline{I}(z) \tag{5.41}$$

The equation (5.41) for the progressive voltage and current wave becomes:

$$-\frac{d\underline{U}^{+}}{dz}(z)\overset{(5.38)}{=}\underline{U}_0^{+}\,\gamma\,e^{\gamma z}=\left(R'+j\omega L'\right)\underline{I}_0^{+}\,e^{\gamma z}\Rightarrow\underline{U}_0^{+}=\frac{\left(R'+j\omega L'\right)}{\gamma}\underline{I}_0^{+} \tag{5.42}$$

By inserting (5.42) into (5.41) the characteristic impedance of a transmission line can be written as follows:

$$\underline{Z}_0=\frac{\underline{U}_0^{+}}{\underline{I}_0^{+}}\overset{(5.42)}{=}\frac{\left(R'+j\omega L'\right)}{\gamma}\overset{(5.35)}{=}\sqrt{\frac{\left(R'+j\omega L'\right)}{\left(G'+j\omega C'\right)}}=R_0+jX_0 \tag{5.43}$$

A transmission line is considered to be a loss-free line if the conductors of the line are perfect ($\sigma\to\infty$) and the surrounding dielectric medium is an ideal dielectric ($\sigma=0$). Thus the resistance of the conductors and the conductance of the surrounding space are zero and the characteristic impedance of such a line can be written as follows:

$$R'=G'=0\Rightarrow Z_0=\sqrt{\frac{L'}{C'}} \tag{5.44}$$

It is interesting now to analyze the transmission line terminated by the load with the impedance $\underline{Z}_L$ as shown in Figure 5.5.

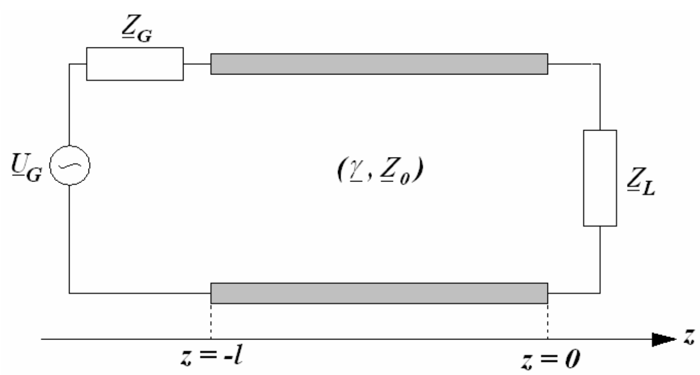


Figure 5.5. The transmission line of the length *l* characterized by the propagation constant γ and the characteristic impedance $\underline{Z}_0$ with the voltage generator (U_G, Z_G) on one side and the load (Z_L) on the another side is shown.

When the generator of the voltage $\underline{U}_G$ and the self-impedance $\underline{Z}_G$ is connected to the transmission line loaded with the impedance $\underline{Z}_L$ the voltage an current waves will propagate along the line from the generator towards the load. Depending on the load impedance the incoming progressive voltage wave will be partially reflected at the end of the line. Therefore along the line the voltage is equal to the superposition of the progressive voltage wave produced by the generator and the regressive reflected wave produced at the transmission line termination point, i.e. at the load:

$$\underline{U}(z) = \underline{U}_0^+ \, e^{-\gamma z} + \underline{U}_0^- e^{\gamma z} \tag{5.45}$$

It is possible to quantify the reflection of the voltage wave by introducing the *voltage reflection coefficient* Γ_L (at the load) [8]:

$$\Gamma_L = \frac{U_0^-}{U_0^+} \tag{5.46}$$

Thus the voltage wave (5.45) can be described by using the reflection (5.46) as follows:

$$\underline{U}(z) = \underline{U}_0^+ \left(e^{-\gamma z} + \Gamma_L \, e^{\gamma z} \right) \tag{5.47}$$

Accordingly, the current wave can be written as follows:

$$\underline{I}(z) = \frac{\underline{U}_0^+}{\underline{Z}_0} \left(e^{-\gamma z} - \Gamma_L \, e^{\gamma z} \right) \tag{5.48}$$

At the load, i.e. at $z = 0$ the load impedance must be equal to the ratio of the voltage and current wave:

$$\underline{Z}_L = \frac{\underline{U}(0)}{\underline{I}(0)} = \frac{\underline{U}_0^+ \left(1 + \Gamma_L\right)}{\frac{\underline{U}_0^+}{\underline{Z}_0} \left(1 - \Gamma_L\right)} = \underline{Z}_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{5.49}$$

Thus, the voltage reflection coefficient becomes:

$$\Gamma_L = \frac{\underline{Z}_L - \underline{Z}_0}{\underline{Z}_L + \underline{Z}_0} \tag{5.50}$$

Apparently, when the transmission line is terminated by the characteristic impedance of the waveguide the reflection is zero, and this is the most desired case:

$$\underline{Z}_L = \underline{Z}_0 \quad \Rightarrow \quad \Gamma_L = 0 \tag{5.51}$$

In this case of the ideal termination the incident power is fully absorbed by the load.

5.5. Scattering Matrix

To make analysis of the transmission lines networks easier, one can consider a certain part of the network as the black-box connected with the remaining part of the network by the so-called ports. The behaviour of our black-box have to be described in the electromagnetic sense. This can be elegantly done in terms of incident and reflected electromagnetic waves, i.e. by using the so-called scattering matrix or s-parameters.

Let us consider the two-port network shown in Figure 5.6. In this figure a_1, a_2 represent the incident and b_1, b_2 represent the reflected voltage waves. The reflected voltage waves can be expressed in terms of the incident waves by using the s-parameters as follows:

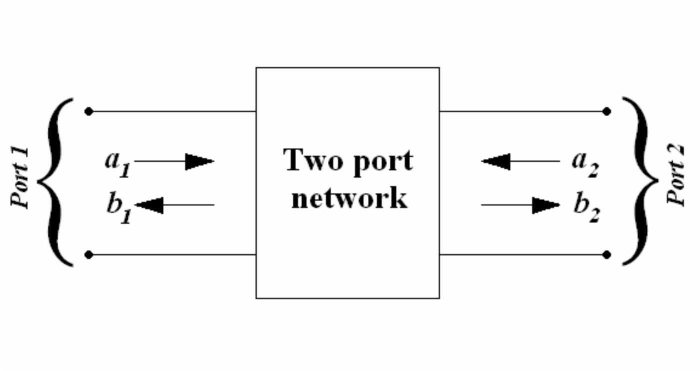


Figure 5.6. Two port scattering network is shown; a_1, a_2 represent the incident voltage waves at the port 1 and port 2 respectively; Similarly b_1, b_2 represent the reflected voltage waves.

$$b_1 = s_{11}\, a_1 + s_{12}\, a_2 \tag{5.52}$$

$$b_2 = s_{21}\, a_1 + s_{22}\, a_2 \tag{5.53}$$

Written in matrix form, (5.52) and (5.53) read:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{5.54}$$

The s-parameters s_{11}, s_{12}, s_{21}, s_{22} for a certain two-port network can be measured in the way presented in Figure 5.7.

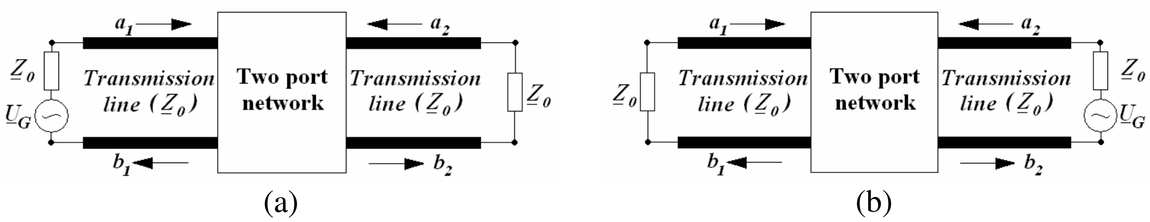


Figure 5.7. The procedure for measurement of the s-parameters is presented; Measurement of s_{11} and s_{21} (a); Measurement of s_{22} and s_{12} (b).

The measurement of the s-parameters is a relatively simple procedure. Namely, if we have a look into the equation (5.52), the parameter s_{11} can be determined as:

$$s_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \tag{5.55}$$

Similarly, from the equation (5.53), the parameter s_{21} reads:

$$s_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \tag{5.56}$$

Apparently, for the measurement of s_{11} and s_{21} the incoming voltage wave at the port 2 a_2 has to be zero. The source-load configuration that guarantees this condition is presented in Figure 5.7a. The input and output port of the network are connected over the transmission lines with the characteristic impedance $\underline{Z}_0$ to the voltage generator and the load. In order to suppress the reflections of the waves from the connection between the input transmission line and the source, the source impedance is chosen to be equal to the characteristic impedance of the transmission line. For the same reason the load is chosen to be equal to the characteristic impedance of the output transmission line. Therefore the incoming wave a_2 at the port 2 which can be produced only as a reflected wave at load must be equal to zero (see the equation (5.51)). By using similar approach presented in Figure 5.7b one can determine the parameters s_{22} and s_{12}:

$$s_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} \tag{5.57}$$

$$s_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \tag{5.58}$$

To make s-parameters more appropriate for practical applications the incident a_1, a_2 and reflected b_1, b_2 voltage waves are normalized in such a way that they represent the square root of the corresponding power:

$$a_1 = \sqrt{\frac{U_{1I}^{\ 2}}{Z_0}} = \frac{U_{1I}}{\sqrt{Z_0}},\ b_1 = \sqrt{\frac{U_{1R}^{\ 2}}{Z_0}} = \frac{U_{1R}}{\sqrt{Z_0}},\ a_2 = \sqrt{\frac{U_{2I}^{\ 2}}{Z_0}} = \frac{U_{2I}}{\sqrt{Z_0}},\ b_2 = \sqrt{\frac{U_{2R}^{\ 2}}{Z_0}} = \frac{U_{2R}}{\sqrt{Z_0}} \tag{5.59}$$

where U_{1I} is the incident voltage at the port 1, U_{1R} is the reflected voltage at the port 1, U_{2I} is the incident voltage at the port 2, and U_{2R} is the reflected voltage at the port 2. Thus it is obvious the following [11]:

$$|a_1|^2 = \text{Incident Power Into Port 1} \tag{5.60}$$

$$|a_2|^2 = \text{Incident Power Into Port 2} \tag{5.61}$$

$$|b_1|^2 = \text{Reflected Power From Port 1} \tag{5.62}$$

$$|b_2|^2 = \text{Reflected Power From Port 2} \tag{5.63}$$

Having (5.60 - 5.63) we can from the definitions (5.55-5.58) draw very important conclusions about the meaning of the scattering parameters [11]:

$$|s_{11}|^2 = \frac{\text{Reflected Power From Port1}}{\text{Incident Power Into Port1}} \tag{5.64}$$

$$|s_{21}|^2 = \frac{\text{Transfered Power From Port1 to Port2}}{\text{Incident Power Into Port1}} = \text{Insertion Power Gain} \tag{5.65}$$

$$|s_{22}|^2 = \frac{\text{Reflected Power From Port2}}{\text{Incident Power Into Port2}} \tag{5.66}$$

$$|s_{12}|^2 = \frac{\text{Transferred Power From Port2 to Port1}}{\text{Incident Power Into Port2}} = \text{Reverse Insertion Power Gain} \tag{5.67}$$

Therefore, the coefficients s_{11} and s_{22} are called *reflection coefficients* at the port 1 and port 2 respectively and the coefficients s_{21} and s_{12} are called *transmission coefficients* [9].

The normalized incident voltage waves a_1, a_2 and normalized reflected voltage waves b_1, b_2 can be written in terms of the voltage and current at the input and output port respectively in the following way:

$$U_1 = U_{1I} + U_{1R} \overset{(5.59)}{=} \sqrt{Z_0}\left(a_1 + b_1\right) \tag{5.68}$$

$$I_1 = I_{1I} + I_{1R} \overset{(5.59)}{=} \frac{1}{\sqrt{Z_0}}\left(a_1 - b_1\right) \tag{5.69}$$

It is very easy to transform the equations (5.68) and (5.69) into the following form:

$$a_1 = \frac{U_1 + Z_0 I_1}{2\sqrt{Z_0}},\ b_1 = \frac{U_1 - Z_0 I_1}{2\sqrt{Z_0}} \tag{5.70}$$

By following the same procedure for the port 2 we obtain:

$$a_2 = \frac{U_2 + Z_0 I_2}{2\sqrt{Z_0}},\ b_2 = \frac{U_2 - Z_0 I_2}{2\sqrt{Z_0}} \tag{5.71}$$

Having the equations (5.70) and (5.71) we can express the reflection coefficients in terms of the two-port network impedance seen from the port 1 ($Z_1 = U_1/I_1$) and port 2 ($Z_2 = U_2/I_2$) as follows:

$$s_{11} = \frac{b_1}{a_1} \overset{(5.70)}{=} \frac{U_1 - Z_0 I_1}{U_1 + Z_0 I_1} = \frac{Z_1 - Z_0}{Z_1 + Z_0} \overset{(5.50)}{=} \Gamma_1 \tag{5.72}$$

$$s_{22} = \frac{b_2}{a_2} \overset{(5.70)}{=} \frac{U_2 - Z_0 I_2}{U_2 + Z_0 I_2} = \frac{Z_2 - Z_0}{Z_2 + Z_0} \overset{(5.50)}{=} \Gamma_2 \tag{5.73}$$

Apparently, the equations (5.72) and (5.73) are the prove that the s-parameters s_{11} and s_{22} are equal to the corresponding voltage reflection coefficients.

5.6. Extraction of Lumped Parameters from S-parameters

Modern analysis of microwave and optical components is based on the numerical filed computation. Afterwards, the field has been used for the scattering matrix computation. Having s-parameters, the corresponding lumped parameters of the equivalent circuit can be extracted. The equivalent circuit of a certain component is then used as a simple part of the complex schematic diagram of the entire device or system. In this section we will present a complete workflow of the extraction of lumped parameters from the field solution for a simple structure of a tiny iris inserted into a parallel plane waveguide. The geometry of the waveguide with the inserted iris is given in Figure 5.8a.

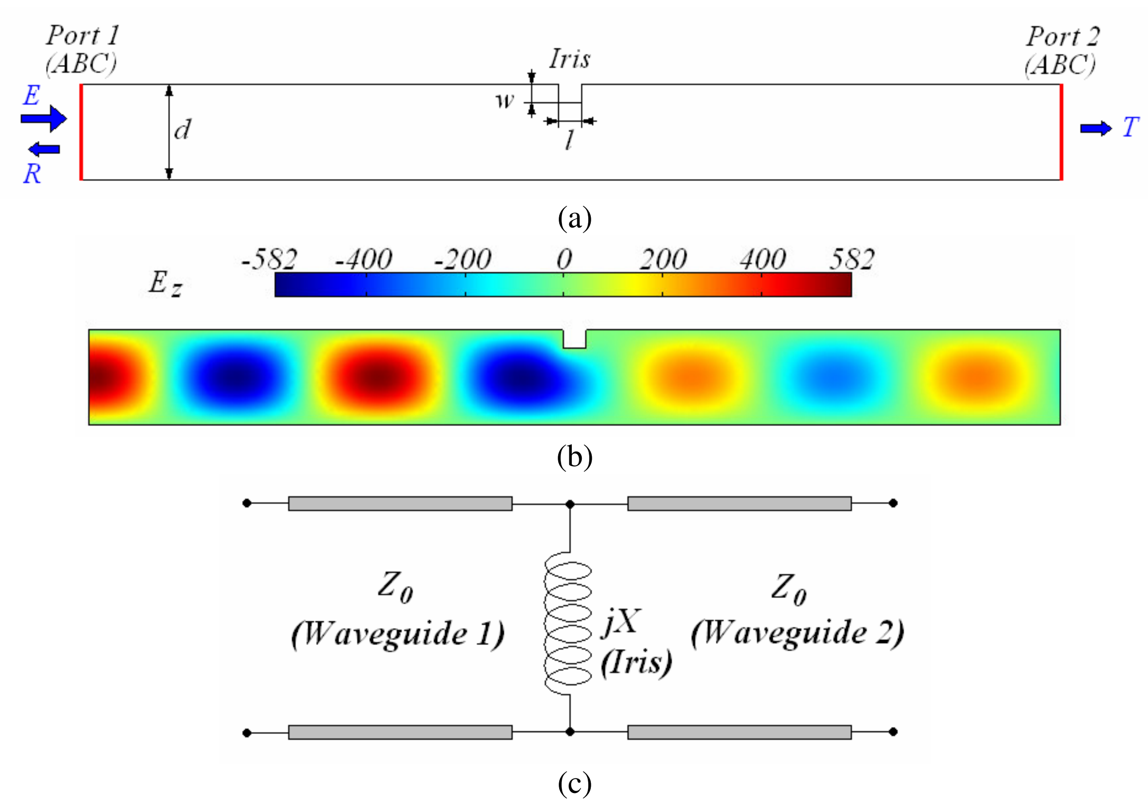


Figure 5.8. The geometry of the tiny iris inserted into the parallel plate waveguide is presented (a); The z-component of the electric field is presented (the fundamental even mode is excited in the waveguide) (b); The dimensions of the waveguide and the iris are *d=0.02m*, *l=0.005m*, *w=0.004m*; The circuit representation with the equivalent lumped parameter of the iris (single inductance $X = \omega L$) is shown.

We will analyze the iris for the fundamental even mode of the parallel plate waveguide. Waveguide ports have been terminated using previously described absorbing boundary conditions (ABC). More information about ABC can be found in Section 5.4. After the electromagnetic field has been calculated, the s-parameters have to be computed. According to the mode matching theory [10] the s-parameters of the waveguide discontinuity can be calculated as follows:

$$S_{11} = \frac{\int\limits_{\partial_2\Omega} \left(E_z - E_{1z}\right) \cdot E_{1z}\, dl}{\int\limits_{\partial_2\Omega} E_{1z} \cdot E_{1z}\, dl} \qquad (5.74)$$

$$S_{21} = \frac{\int\limits_{\partial_3\Omega} E_z \cdot E_{2z}\, dl}{\int\limits_{\partial_3\Omega} E_{2z} \cdot E_{2z}\, dl} \qquad (5.75)$$

where E_{1z} , and E_{2z} are the forms of the excitation field at the input and output port respectively and E_z is the electric field obtained as a FEM solution of our problem. Due to the problem symmetry and the absence of the losses we will have $s_{22} = s_{11}$ and $s_{21} = s_{12}$. For geometry given in Figure 5.8a and the frequency of the fundamental even mode of $f = 9 \cdot 10^9 Hz$ the s-parameters have the following values:

$$|S_{11}|^2 = 0.349\,,\ |S_{21}|^2 = 0.651 \tag{5.76}$$

$$|S_{11}|^2 + |S_{21}|^2 = 1.000 \tag{5.77}$$

According to the power conservation law (5.77) our results are very accurate. Since our structure including the iris is perfectly conductive (loss free) the iris can be approximated with a single reactance (if the iris is tiny enough). This reactance can be determined from the s-parameter s_{11} by using the equation (5.73) in the following way:

$$s_{11} \overset{(5.73)}{=} \frac{jX - Z_0}{jX + Z_0} \quad \Rightarrow \quad X = Z_0 \frac{1+s_{11}}{1-s_{11}} \tag{5.78}$$

where Z_0 is the characteristic impedance of the parallel plate waveguide:

$$Z_0 = \frac{E_z}{H_y} = \frac{\gamma}{\omega\mu} = \frac{\sqrt{\omega^2\mu\varepsilon - (\pi/d)^2}}{\omega\mu} = 681\Omega \tag{5.79}$$

Thus the equivalent inductance of the iris reads:

$$X = Z_0 \frac{1+s_{11}}{1-s_{11}} = 2649\Omega \tag{5.80}$$

This approach is widely used for the analysis of the microwave and optical components (including MEMS). The field analysis, as an initial step, allows us to compute the s-parameters. Having the s-parameters we can extract the equivalent lumped parameters and include them into a complex schematic circuit representation of the entire device. This device as a system can be afterwards analyzed by solving only the equations of the circuit theory.

5.7. References

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