Exercise 2

Question:

The key element of a Xerox machine is a charged drum which is coated with photoconductive material. A simplified sketch of this element is given in fig. 1. Calculate the electric field between the two electrodes.

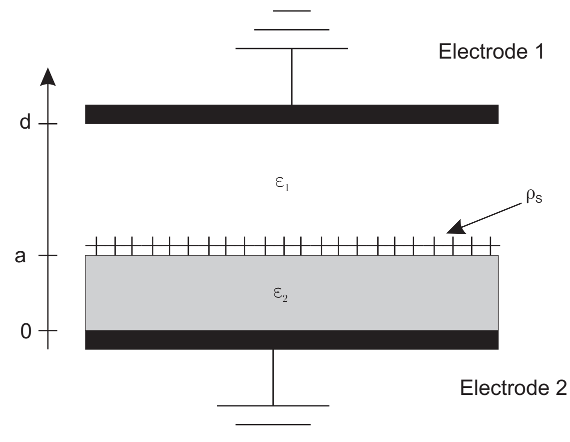

Figure 1: Simplified picture of a xerox machine drum coated with photoconductor. The photoconductor carries a positive charge ρ_S.

Solution:

Firstly the domain is split into two half domains at the surface of the photoconductor. Then the governing differential equation is solved in each domain separately:

$$\triangle\Phi = 0 \quad (1)$$

$$\Phi_1(x) = A_1 x + B_1 \quad (2)$$

$$\Phi_2(x) = A_2 x + B_2 \quad (3)$$

Subsequently, the constants are determined by applying the boundary conditions.

$$\Phi_1(d) = 0 = A_1 d + B_1 \quad (4)$$

$$\Phi_2(0) = 0 \rightarrow B_2 = 0 \quad (5)$$

$$\Phi_1(a) = \Phi_2(a) \quad (6)$$

$$D_{1n} - D_{2n} = \rho_s \rightarrow -\varepsilon_1 \frac{d\Phi_1}{dx} + \varepsilon_2 \frac{d\Phi_2}{dx} = \rho_s \quad (7)$$

The constants yield to:

$$A_1 = \frac{-\rho_s}{\varepsilon_1(1 + \frac{\varepsilon_2 d}{\varepsilon_1 a} - \frac{\varepsilon_2}{\varepsilon_1})} \quad (8)$$

$$B_1 = \frac{\rho_s d}{\varepsilon_1(1 + \frac{\varepsilon_2 d}{\varepsilon_1 a} - \frac{\varepsilon_2}{\varepsilon_1})} \quad (9)$$

$$A_2 = \frac{\rho_s(\frac{d}{a} - 1)}{\varepsilon_1(1 + \frac{\varepsilon_2 d}{\varepsilon_1 a} - \frac{\varepsilon_2}{\varepsilon_1})} \quad (10)$$

$$B_2 = 0 \quad (11)$$

The electric field is determined as follows:

$$E_1 = -A_1 \quad (12)$$

$$E_2 = -A_2 \quad (13)$$

Further detailed information is elaborated in M. N. O. Sadiku, "Elements of Electromagnetics", Sounders College Publishing, Orlando, USA, 1989.

Exercise 3

Determine a general potential function V for the region inside the rectangular trough of infinite length in z-direction whose cross section is shown in fig. 2. Furthermore calculate a specific potential V_m at $x = \frac{a}{2}$, $y = \frac{3a}{4}$ for $V_0 = 100\,\mathrm{V}$ and $b = 2a$.

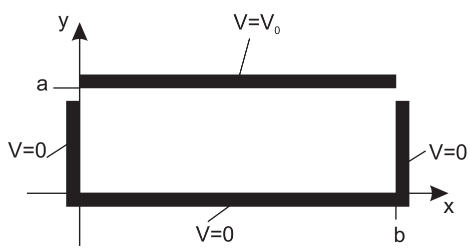

Figure 2: Boundary conditions of the electrical trough.

Solution:

Since the trough has an infinite length in the z-direction the potential V only depends on x and y.

$$\triangle V = \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0 \quad (14)$$

In order to solve this differential equation for the imposed boundary conditions the separation of variables approach is utilised.

$$V(x,y) = X(x)Y(y) \quad (15)$$

Thus the differential equation yields to:

$$X''Y + Y''X = 0 \tag{16}$$

Now we separate to

$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda. \tag{17}$$

As a separation constant a positive real number $\lambda = \beta^2$ is chosen. Other separation constants may not satisfy the given boundary conditions. Thus we solve the differential equation to

$$X'' + \beta^2 X = 0 \tag{18}$$

$$X(x) = g_0 \cos(\beta x) + g_1 \sin(\beta x) \tag{19}$$

Imposing the boundary conditions yields:

$$X(x = 0) = 0 \to g_0 = 0 \tag{20}$$

$$X(x = b) = 0 \to \beta = \frac{n\pi}{b} \tag{21}$$

Thus $X_n(x) = g_n \sin(\frac{n\pi x}{b})$ is the solution for X. Similarly we solve

$$Y'' - \beta^2 Y = 0 \tag{22}$$

$$Y(y) = h_0 \cosh(\beta y) + h_1 \sinh(\beta y) \tag{23}$$

Imposing the boundary conditions yields:

$$Y(y = 0) = 0 \to h_0 = 0 \tag{24}$$

Thus $Y_n(x) = h_n \sinh(\frac{n\pi y}{b})$ is the solution for Y. The solution in x and y is

$$V(x, y) = \sum_n g_n h_n \sinh(\frac{n\pi y}{b}) \sin(\frac{n\pi x}{b}). \tag{25}$$

Now the constant $c_n = g_n h_n$ has to be determined by the boundary condition

$$V(x, y = a) = V_0 = \sum_n c_n \sinh(\frac{n\pi a}{b}) \sin(\frac{n\pi x}{b}). \tag{26}$$

Integrating with test functions $\sin(\frac{m\pi x}{b})$ over $0 < x < b$

$$\int_0^b V_0 \sin(\frac{m\pi x}{b}) dx = \sum_n c_n \sinh(\frac{n\pi a}{b}) \int_0^b \sin(\frac{m\pi x}{b}) \sin(\frac{n\pi x}{b}) dx. \tag{27}$$

With the orthogonality property of the sin function we get

$$\frac{V_0 b}{n\pi}(1 - \cos n\pi) = c_n \sinh(\frac{n\pi a}{b}) \frac{b}{2} \tag{28}$$

This is for n even

$$c_n = 0. \tag{29}$$

For n odd we obtain

$$c_n = \frac{4V_0}{n\pi \sinh(\frac{n\pi a}{b})}. \tag{30}$$

The complete solution is

$$V(x,y) = \frac{4V_0}{\pi} \sum_i \frac{\sin\left(\frac{(2i-1)\pi x}{b}\right) \sinh\left(\frac{(2i-1)\pi y}{b}\right)}{n \sinh\left(\frac{(2i-1)\pi a}{b}\right)}. \tag{31}$$

Finally the specific potential V_m is determined to $V_m = 68.14\,\mathrm{V}$ for a potential $V_0 = 100\,\mathrm{V}$, $b = 2a$ at $x = \frac{a}{2}$, $y = \frac{3a}{4}$.

Exercise 4

An electromagnetic wave described by its electric field as

$$\vec{E} = 4\sin(2\pi 10^7 t - 0.8x)\vec{e}_z \tag{32}$$

is propagated in a linear, nonmagnetic, homogeneous and isotropic medium. Determine the relative permittivity ε_r and the wave impedance $Z = \sqrt{\frac{\mu}{\varepsilon}}$ of the medium. Furthermore calculate the time averaged power density carried by the wave as well as the total power crossing an area of $100\,\mathrm{cm}^2$ in the plane $2x + y = 5$.

Solution:

Extracting from the given electric field

$$\vec{E} = 4\sin(2\pi 10^7 t - 0.8x)\vec{e}_z = E_0 \sin(\omega t - \beta x)\vec{e}_z \tag{33}$$

equation the following parameter:

$$\begin{aligned} \omega &= 2\pi 10^7 \\ \beta &= 0.8 \\ \mu &= \mu_0 \\ \varepsilon &= \varepsilon_r \varepsilon_0 \end{aligned}$$

Starting with

$$\begin{aligned} \beta &= \omega\sqrt{\mu_0 \varepsilon_r \varepsilon_0} \\ \varepsilon_r &= \frac{\beta^2}{\omega^2 \varepsilon_0 \mu_0} \end{aligned} \tag{34}$$

the relative permittivity ε_r is determined to $\varepsilon_r = \frac{12}{\pi} = 14.59$ and the wave impedance respectively by the following relation:

$$Z = \sqrt{\frac{\mu_0}{\varepsilon}} = 10\pi^2 = 98.7\Omega. \tag{35}$$

The Pointing vector is defined as

$$\vec{S} = \vec{E} \times \vec{H}. \tag{36}$$

The magnetic field strength has to be derived first. We start from Faraday's law

$$\begin{aligned} \nabla \times \vec{E} &= -\mu_0 \frac{d\vec{H}}{dt} \\ \vec{e_y}\frac{dE}{dx} &= E_0\beta\cos(\omega t - \beta x)\vec{e_y} \\ \vec{H} &= E_0\frac{\beta}{\mu_0}\vec{e_y}\int\cos(\omega t - \beta x)\,dt \\ \vec{H} &= -E_0\frac{\beta}{\omega\mu_0}\sin(\omega t - \beta z)\vec{e_y} \end{aligned}$$

and utilising the relation

$$\frac{\beta}{\omega\mu_0} = \frac{\omega\sqrt{\mu_0\varepsilon}}{\omega\mu_0} = \sqrt{\frac{\varepsilon}{\mu_0}} = \frac{1}{Z}$$

the magnetic fields yield to

$$\vec{H} = -\frac{E_0}{Z}\sin(\omega t - \beta x)\vec{e}_y = \frac{E}{Z}\vec{e}_y. \tag{37}$$

Consequently the pointing vector yield to

$$\vec{S} = \frac{E_0^2}{Z}\sin^2(\omega t - \beta x)\vec{e}_x. \tag{38}$$

and finally the time averaged pointing vector can be described as

$$\vec{S}_{ave} = \vec{e}_x\frac{1}{T}\int_0^T \frac{E_0^2}{Z}\sin^2(\omega t - \beta x)\,dt = 81.05\vec{e}_x\,\frac{\text{mW}}{\text{m}^2}, \tag{39}$$

where $T = \frac{2\pi}{\omega}$ describes the time period. The total power crossing the plane $2x + y = 5$ with an angle $\varphi = \arctan(\frac{1}{2}))$ with respect to its normal vector $\vec{n} = (2, 1, 0)^T/\sqrt{5}$, finally yields to:

$$\begin{aligned} P_{avg} &= \vec{S}_{ave}\cdot\vec{A} = \vec{S}_{ave}\cdot\vec{n}A \\ &= \vec{e}_x\cdot\vec{n}\,A\,S_{ave} \\ &= \cos(\arctan(\tfrac{1}{2}))\,A\,S_{ave} = 724.5\,\mu\text{W}. \end{aligned} \tag{40}$$